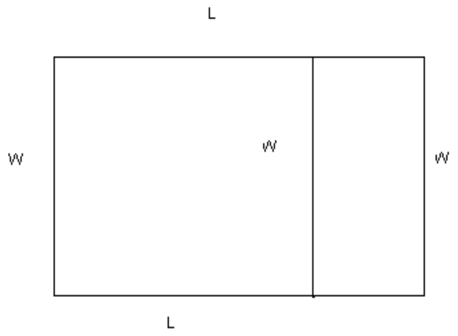
An Applied Optimization Problem

Problem: A farmer person needs to fence off a 1.5 million square feet rectangle with a single interior partition parallel to one of the sides (to keep the girl cows away from the boy cows who are just ordinary run of the mill boy cows trying to run a mill). See the accompanying sketch. What dimensions will minimize the cost to the farmer person?

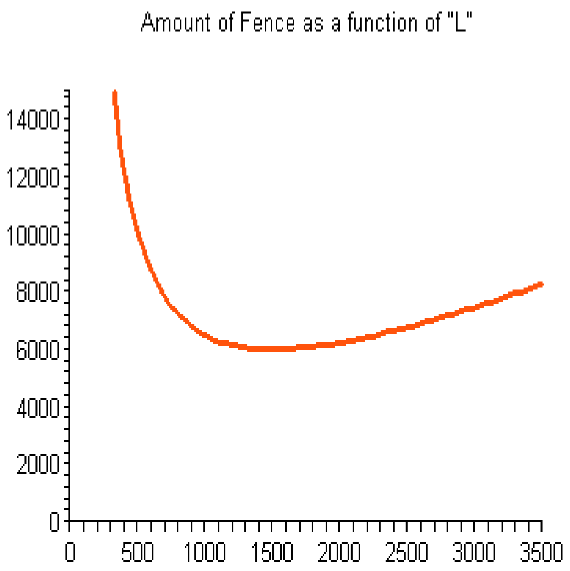


Solution You first have to realize what is being asked. We want to minimize the cost of the fence which will be done if we can minimize the **amount** of fence required. If A is the amount of fencing required then we know that $A = 3W + 2L$.

The problem with this expression is that it contains too many variables. We can eliminate one of them by using the area constraint: the area must be 1.5×10^6 sq. ft so $W \cdot L = 1.5\times 10^6$ so we can write that $W = \dfrac{1.5\times 10^6}{L}$ and from this we can write the amount of fence as a function of L alone:

$$A(L) = 2L + 3\cdot\frac{1.5\times 10^6}{L} = 2L + \frac{4.5\times 10^6}{L}.$$

Before doing anything further, once you have a function that models the particular situation you should state the domain of the model. In this case, L must be positive and 0 is not allowed so that $D_A = (0,\infty)$. A Maple generated plot of $A = A(L)$ is presented to the right. From the graph you can see that there is a value of L that will produce a minimum value for the amount of fence. From the graph we might guess that the value is around $L = 1500$



Now comes the calculus. To minimize A we look for the extreme points by lookin at the critical numbers of AI.

$$A'(L) = 2 - \frac{4.5\times 10^6}{L^2}\ .\ A'(L) = 0 \Leftrightarrow 2L^2 - 4.5\times 10^6 = 0 \text{ so that } L = \pm\sqrt{\frac{4.5\times 10^6}{2}}$$

but we discard the negative value since $L > 0$ always. There are no other critical numbers since even though A' does not exist if $L = 0$, $0 \notin D_A$. All that remains is to

prove that $L = \sqrt{\frac{4.5\times10^6}{2}} = 1500$ does, in fact, correspond to a minimum value for the function. This is done by s sign table for A' over the intervals determined by its "key numbers": 0 and 1500.

x	$A'(L)$	$A(L)$
(0,1500)	-	$\downarrow 'ing$
$(1500,\infty)$	+	$\uparrow 'ing$

The table makes it clear that the absolute minimum value occurs when $L = 1500\ \ ft$. To find the width we use the relationship $W = \frac{1.5\times10^6}{L}$ with $L = 1500\ \ ft$ to get $W = 1000\ \ ft$. This confirms the impression gained from the graph.