Analyzing a Function Completely – An Example

Let $f(x) = 3x^5 - 5x^3 + 3$. To analyze a function completely (that is, as completely as we can with the tools of our course) you must consider several questions. You should try to always follow the same systematic approach.

1. Determine the domain of *f*.

This will help you avoid embarrassments such as trying to draw a graph over an interval for which the function does not exist or "finding" a relative maximum point where the function has no point.

In the example, *f* is a polynomial and so has all **R** as its domain.

2. Determine, if possible, the x- and y-intercepts.

The y-intercept (there can only be one!) is the point $(0, f(0))$ when $0 \in D_f$.

The x-intercepts are found as the solutions of the equation $f(x) = 0$. When this equation is solvable by elementary techniques you are expected to find the intercepts.

In the example, $3x^5 - 5x^3 + 3 = 0$ requires advanced techniques that are well beyond the scope of this course. Using a computer I found that $f(x) = 0$ if $x \approx -1.4$. Since $f(0) = 3$, the y-intercept is (0,3).

3. Determine the existence of horizontal and vertical asymptotes.

If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$ then the line $y = L$ is a horizontal asymptote. A function can have 0, 1 or 2 horizontal asymptotes.

If $\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty$ or if $\lim_{x\to a^-} f(x) = \infty \text{ or } -\infty$ or if $\lim_{x\to a} f(x) = \infty \text{ or } -\infty$ then the vertical line $x = a$ is a vertical asymptote of *f*. You look for vertical asymptotes where the function is discontinuous (which is *usually* where any of the denominators are zero). For each candidate for a vertical asymptote you must show the two limit calculations.

In the example, since *f* is a polynomial it has no asymptotes of either kind and, in this case, you would not be expected to show any work.

4. Determine the intervals of monotonicity of *f* and the Coordinates of the local extrema of *f*.

This is done by determining where $f'(x) > 0$ and where $f'(x) < 0$. To do this, you must determine the sign of f' on each of the intervals determined by the "key numbers" of f': the numbers where $f'(x) = 0$ or where f' does not exist.

$f'(x) = 15x^4 - 15x^2 = 15x^2(x-1)(x+1) = 0 \Leftrightarrow x \in \{0, \pm 1\}$ so there are three "key numbers" of f' (which, in this e.g., are also the critical numbers of *f*). Now, we can make a sign table to determine the sign of f' on the intervals determined by its key numbers and hence find out where *f* is increasing/decreasing.

x	$f'(x)$	$f(x)$
$(-\infty, -1)$	+	$\uparrow$ '*ing*
$(-1, 0)$	-	$\downarrow$ '*ing*
$(0,1)$	-	$\downarrow$ '*ing*
$(1, \infty)$	+	$\uparrow$ '*ing*

From the table you can deduce that there is a local **maximum point** at $(-1, 5)$ and a **local minimum** at (1,1) and that there is no relative extremum when x = 0. It shows that *f* is increasing on $(-\infty, -1) \cup (1, \infty)$ and decreasing on $(-1, 1)$.

5. <u>Determine the Intervals where *f* is Concave Up or Down and the Coordinates of any Inflection Points</u>.

This is down by determining where $f''(x) > 0$ and where $f''(x) < 0$ positive and negative. Therefore, we make a sign table for f'' on the intervals determined by its key numbers.

Since $f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$ the key numbers are $x \in \left\{0, \pm\frac{1}{\sqrt{2}}\right\}$. This produces the following table.

x	$f''(x)$	$f(x)$
$(-\infty, -1/\sqrt{2})$	-	$\cap$
$(-1/\sqrt{2}, 0)$	+	$\cup$
$(0, 1/\sqrt{2})$	-	$\cap$
$(1/\sqrt{2}, \infty)$	+	$\cup$

The table indicates that there is an inflection point at $\left(-1/\sqrt{2}, f\left(-1/\sqrt{2}\right)\right)$, $(0,3)$ and $\left(1/\sqrt{2}, f\left(11/\sqrt{2}\right)\right)$. It shows that *f* is concave down on $(-\infty, -1/\sqrt{2}) \cup (0, 1/\sqrt{2})$ and concave up on $(-1/\sqrt{2}, 0) \cup (1/\sqrt{2}, \infty)$.

All that remains now is to make a "rough sketch" of $y = f(x)$. A "**rough sketch**" contains the following elements:

- The x- and y-intercepts (it is not *always* possible to find the x-intercepts exactly)
- The horizontal and vertical asymptotes as dotted lines
- The coordinates of the local/relative extrema
- The coordinates of the inflection points

The only thing that is rough about a "rough sketch" is that absolute scale is not adhered to. A computer generated plot of $f(x) = 3x^5 - 5x^3 + 3$ is supplied below.

DO NOT FOOL YOURSELF, analyzing a function as completely as above and being able to reflect that analysis in a rough sketch takes **PRACTICE**.