

Analyzing a "Real World" Inflection Point or The Law of Diminishing Returns

Suppose a corporation has launched a big-time advertising blitz and theorized (based on past experience, sampling data, etc.) that sales S of a product as a function of the money x spent on advertising (x in thousands of dollars) is given by

$$S = S(x) = \frac{1}{10000}(300x^2 - x^3) \text{ for } x \in [0, 200].$$

A Maple plot of $S = S(x)$ is given below.

Problem: The plot looks like it has an inflection point about half way along. Find the coordinates of it and make recommendations to your boss.

Solution: Inflection points occur where the curvature changes from concave up to down or vice-versa. Hence, we examine the second derivative of S :

$$S'(x) = \frac{1}{10000}(600x - 3x^2)$$

$$S''(x) = \frac{1}{10000}(600 - 6x).$$

The "key numbers" of S'' occur where $600 - 6x = 0$ which is when $x = 100$. To show that this is indeed an inflection point we do a sign table for S'' as follows:

x	S''	S
$(0, 100)$	+	∪
$(100, 200)$	-	∩



The table shows that there is an inflection point at $(100, 200)$. What does it mean?? Before we can answer this we need to remember something from Marginal Analysis: by

definition, $S'(x) = \lim_{\Delta x \rightarrow 0} \frac{S(x + \Delta x) - S(x)}{\Delta x}$. It follows that *if* Δx is "small" then

$$S'(x) \approx \frac{S(x + \Delta x) - S(x)}{\Delta x}.$$

In commerce, it is usual that the smallest practical Δx is 1 (one). This results in the approximation $S'(x) \approx S(x + 1) - S(x)$. And this is what we want. (Aside: it is from this approximation formula that the term "marginal analysis" comes.)

When $x < 100$ we have $S'(x) > 0$ so $S(x)$ is increasing. Therefore, it must be that $S(x+1) - S(x)$ is increasing which means that the difference between $S(x+1)$ and $S(x)$ is increasing. This means that each additional thousand dollars of advertising is producing **more** sales than the previous \$1000 input. Hence, it behooves the powers that be to continue increasing the advertising budget.

When $x > 100$ we have that $S''(x) < 0$ so S' is **decreasing**. Therefore, $S(x+1) - S(x)$ is decreasing and this implies that each additional thousand dollars of advertising is producing **less** than the previous \$1000 dollar increase. The company is now getting less and less "bang for its buck" and should scale back its advertising campaign. Each extra dollar is giving **less and less return**. Once x is past 100, the company is experiencing the effects of the **law of diminishing returns**. Your advice to the boss should now be obvious.

The *law of diminishing returns* is familiar to anyone with an older car. Sooner or later you realize that each dollar spent in repairs is producing less and less result and finally you have to "cut your losses" and scrap the old heap.