Factoring Polynomials with Integer Coefficients

Definition 1. A **polynomial** function is any function of the form $\boxed{p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0}$ where a_i is a number (coefficient) and $n \in \mathbb{N} = \{0,1,2,3,\ldots\}$. The **degree** of a polynomial is the highest power – the power of the "leading term."

Example 1. $p(x) = 7x^6 - \frac{1}{2}x^3 + \sqrt{2}$ and $q(x) = x^2 - x + 1$ are polynomials of degrees 6 and 2 respectively. Linear functions are polynomials of degree 1 and constant functions are polynomials of degree 0.

For <u>all that follows</u>, we will assume that all the coefficients are **integers**.

Definition 2. If a is a number and $p(a) = 0$ then a is called a **root** or **zero** of $p(x)$.

Example 2. If $p(x) = 2x^3 + 5x^2 + x - 2$ then $p(-2) = -16 + 20 - 4 = 0$ so –2 is a root of *p*.

Factor Theorem. If a is a root of $p(x)$ then $x - a$ is a factor of $p(x)$.

Example 3. $x - (-2) = x + 2$ is a factor of $p(x) = 2x^3 + 5x^2 + x - 2$ since $p(-2) = 0$. Knowing this, you can use **long division** to divide $p(x)$ by $x + 2$ to get $p(x) = (x+2)\left(2x^2 + x - 1\right)$ (do it!!). You can then factor $2x^2 + x - 1$ by inspection and obtain that the **complete factorization** of *p* is $p(x) = (x+2)(2x-1)(x+1)$. From this you can quickly see that $p(x) = 0$ when $x = -2,\ x = \frac{1}{2}$ or $x = -1$.

Rational Roots Theorem. If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$ has a root of the form $\frac{a}{b}$ then a must be a divisor of a_0 and b must be a divisor of a_n.

Example 3. If $p(x) = 3x^3 + 20x^2 + 23x - 10$ has any roots of the form $\frac{a}{b}$ then it must be that $a \in \{\pm1, \pm2, \pm5, \pm10\}$ and $b \in \{\pm1, \pm3\}$. It follows that ***if*** there are any rational roots of *p* then they must be amongst the numbers $\left\{\pm1, \pm2, \pm5, \pm10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}\right\}$.
The only way to tell if any of the "candidates" are actually roots of *p* is by **trial and error** (i.e. by "plugging in"). It should be said that if none of the candidates work then you know that the polynomial in question has *no rational roots*.

Once you have found a root of *p* using the **rational roots theorem** then you can use long division to write *p* in a factored form and then continue using the theorem and division or trial and error to factor *p* as completely as possible.

Example 4. Show that $f(x) = x^2 - 2$ has no rational roots.

Solution. By the theorem, if *f* has a rational root then it must be one of $\{\pm 1, \pm 2\}$. By plugging in each of these into *f* we see that none of them give zero so it follows that *f* has no rational roots. Now, since we know that $x^2 - 2 = 0 \Leftrightarrow x = \pm\sqrt{2}$ it follows that $\sqrt{2}$ is *not* a rational number (i.e. not a fraction).