Inverse Functions

Definition 1: a function *f* is called **one-to-one** (abbreviated as 1-1) if whenever $f(a) = f(b)$ it must be that $a = b$.

What this *really* says is that each *y* value in the **range of *f*** (abbreviated R_f) has only one corresponding *x* value in the **domain of *f*** (abbreviated D_f).

Example 1. Show that $f(x) = x^2$ is *not* 1-1.

Solution. $f(-2) = 4 = f(2)$ but $-2 \neq 2$ so that *f* fails to meet the criteria of the definition. If you draw a horizontal line through the graph of $y = x^2$ at level $y = 4$ you can see that the line passes through the graph twice (once at $x = -2$ and again at $x = 2$) and this shows geometrically that for the number 4 in R_f there is *more than* one corresponding value in D_f. Think of the **horizontal line test** you learned in High School.

Example 2. Show that $f(x) = 2x - 4$ is 1-1.

Solution. Suppose that there are numbers *a* and *b* such that $f(a) = f(b)$.:

$$\begin{aligned} &f(a) = f(b) \\ &2a - 4 = 2b - 4 \\ &2a = 2b \\ &a = b \end{aligned}$$

So, for this *f*, if $f(a) = f(b)$ it *must* follow that $a = b$ and so *f* is 1-1 (draw the graph of $y = 2x - 4$ and do the horizontal line test).

The reason for considering 1-1 functions is because only 1-1 functions have what are called an **inverse**. The **inverse of *f*** is denoted f^{-1} and it has the property that if $f(a) = b$ then $f^{-1}(b) = a$. In other words, f^{-1} reverses whatever *f* does. You can only do this properly with 1-1 functions.

To see why $f(x) = x^2$, which is *not* 1-1, does not have an inverse suppose that $f(a) = 4$. Then how do you define f^{-1}? Does $f^{-1}(4) = 2$ or does $f^{-1}(4) = -2$? Remember, a function can have only *one* *y*-value for each *x*-value.

Definition 2. If *f* is a 1-1 function with domain D_f and range R_f then the **inverse of *f***, denoted f^{-1}, is a function defined by the rule $f(x) = y$ if and only if $f^{-1}(y) = x$ for each $x \in D_f$ and for each $y \in R_f$.

In this case, if $x \in D_f$ then $f^{-1}\left(f(x)\right) = f^{-1}(y) = x$ and $f\left(f^{-1}(y)\right) = f(x) = y$ for each $y \in R_f = D_{f^{-1}}$. These are sometimes called the **cancellation equations** of inverse functions: $\boxed{f\left(f^{-1}(x)\right) = x \ \forall \ x \in D_f}$ and $\boxed{f^{-1}\left(f(x)\right) = x \ \forall \ x \in D_{f^{-1}}}$.

Sometimes the definition makes it possible to find an explicit algebraic formula for f^{-1} as in the following example.

Example 3. Use the definition to find the inverse of $f(x) = 2x - 4$.

Solution. We have already showed that *f* is 1-1 so we know that it does have an inverse. Let $y = f^{-1}(x)$ and proceed as below.

$$\begin{aligned}
& y = f^{-1}(x) \\
& f(y) = x && \text{(by definition)} \\
& 2y - 4 = x \\
& y = \frac{1}{2}x + 2 && \text{(by ordinary algebra)} \\
& f^{-1}(x) = \frac{1}{2}x + 2
\end{aligned}$$

Notice that $f(3) = 2$ and that $f^{-1}(2) = \frac{1}{2} \cdot 2 + 2 = 3$. You should confirm that $f\left(f^{-1}(x)\right) = x \ \forall \ x$ and that $f^{-1}\left(f(x)\right) = x \ \forall \ x$ since that would **prove** that the answer is correct.

Very often you cannot find a "nice" formula for f^{-1} but must rely solely on the definition and your knowledge of the original function as seen in the next example.

Example 4. Let $f(x) = \sin x$ for $x \in [-\pi/2, \pi/2]$ and define $f^{-1}(x) = \sin^{-1} x$ by saying $\boxed{\sin x = y \text{ if and only if } \sin^{-1} y = x}$. It follows from this definition that $\sin\left(\sin^{-1}(y)\right) = y \ \forall \ y \in D_{\sin^{-1}}$ and that $\sin^{-1}\left(\sin(x)\right) = x \ \forall \ x \in D_{\sin}$ so the cancellation equations are satisfied. Can you figure out the domain and range of $\sin^{-1}$?

What, for example, is the value of $\sin^{-1}\left(\frac{1}{2}\right)$? To find it, note that $w = \sin^{-1}\left(\frac{1}{2}\right) \Leftrightarrow \sin w = \frac{1}{2}$. This means that we are looking for the **number** between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (or the **angle** between -90° and 90°) whose sine value is $\frac{1}{2}$. You can check

(I hope!!) that $\sin\left(\frac{\pi}{3}\right)=\frac{1}{2}$ and so we know now that $\sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$ (or, if you are working with angles, then $\sin^{-1}\left(\frac{1}{2}\right)=60^{\circ}$). Try and use this thinking to find the common values for the arcsin function. (Arcsin is another, older, name for $\sin^{-1}$.)

The **textbook** covers general **inverse functions** in section **1.6** on pages 63 – 67 and **inverse trigonometric functions** on pages 72 – 74.