Differentiability And Continuity

Theorem: If $y = f(x)$ is *differentiable* at x = a then f is continuous at x = a.

Recall:

f is differentiable at x = a means that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists (and is called f′(a)). Also, f is continuous at x = a if and only if $\lim_{x\to a} f(x) = f(a)$.

Proof:

$$f(x)-f(a) = f(x)-f(a) \text{ (wow!)}$$
$$= \frac{f(x)-f(a)}{x-a}\cdot(x-a) \text{ if } x \neq a$$
$$\therefore \lim_{x\to a}\left(f(x)-f(a)\right) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right]$$
$$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = \lim_{x\to a}\frac{f(x)-(a)}{x-a}\lim_{x\to a}(x-a)$$
$$\lim_{x\to a} f(x) - \lim_{x\to a} f(a) = f'(a)\cdot 0 = 0 \text{ so it follows that}$$
$$\lim_{x\to a} f(x) = \lim_{x\to a} f(a) = f(a) \text{ (since } f(a) \text{ is a number)}$$

Therefore, f is continuous at x = a.

Q.E.D.

So, differentiability implies continuity but it is NOT true that continuity implies differentiability.

Example: $f(x) = |x|$ is not differentiable at (0,0).

Proof: By definition, $f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$. Now since, $y = |x|$ is a piecewise defined function that changes definition at x = 0 we have to consider one-sided limits when approaching 0 (if x were approaching any other number in this e.g. you would NOT need to use one-sided limits).

$\lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = \lim_{x\to 0^+} 1 = 1$. BUT,

$\lim_{x\to 0^-}\frac{|x|}{x}=\lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}(-1)=-1\neq\lim_{x\to 0^+}\frac{|x|}{x}$ so it follows that the limit DOES NOT EXIST and from that you conclude that $f(x)=|x|$ is not differentiable at $x=0$. (Notice that there is no problem with this function anywhere else in terms of differentiability). Because $f(x)=|x|$ is continuous at (0,0) it follows that continuity does NOT imply differentiability.