<u>An Example of *Simple Harmonic Motion*</u>

A spring that stretches in *proportion* to an applied force is obeying **Hooke's Law.**

Hooke's Law: If a mass *m* is attached to a spring and the mass is free to move on a horizontal, frictionless surface then the spring exerts a force on the mass given by $F = -kx$ when the mass is displaced a "small" distance *x* from its equilibrium (resting) position.

F is called a *linear restoring force* since it is linearly proportional to the displacement and is always directed toward the equilibrium position. (It "tries" to restore the equilibrium position.)

By Newton's 2nd Law of Motion[1], $F = ma = -kx$ where a is the acceleration. Therefore, $a = -\frac{k}{m}x$ (so acceleration is proportional to the displacement from equilibrium and is in the opposite direction).

Suppose the mass is displaced a (maximum) distance of A units from equilibrium and then released, so its initial acceleration is $a = -\frac{kA}{m}$. It will travel through the equilibrium position (x = 0) until it reaches a position of $-A$ units from equilibrium, at which time its acceleration will be $a = -\frac{k(-A)}{m} = \frac{kA}{m}$ (at which instant its velocity will be zero). In one full cycle of motion it will travel 4A units and the motion will repeat.

Now, since acceleration is the second derivative of the position function, we have that a spring obeying Hooke's Law must satisfy the *differential equation*:
$\frac{d^2x}{dt^2} = \frac{-k}{m}x = -v^2x$ where $w^2 = \frac{k}{m}$

Soon into your *first* course in *Differential Equations* at university you will learn how to solve such equations and be able to obtain the general solution for yourself:

$$x = x(t) = c_1 \sin(wt) + c_2 \cos(wt) \qquad (*)$$

where c_1 and c_2 are constants and $w^2 = \frac{k}{m}$. Furthermore, it can be verified (see problem #2) that $M\cos(wt + d)$ is an equivalent form of (*) where v is as before and d is a constant that can be determined by the *initial conditions*. If we want the initial velocity to be zero and the initial position to be A then $M\cos(w0 + d) = M\cos(d) = A$ and this

---

[1] The acceleration of an object is directly proportional to the resultant force acting on it and inversely proportional to the mass. i.e. $a = \frac{F}{m}$ or $F = ma$.

will be satisfied if we take $d = 0$ and $M = A$. That is, we can represent the motion by the equation $x = A\cos(wt)$.

An object that obeys such an equation is said to be in ***Simple Harmonic Motion.*** The *period* of the motion is easily seen to be $T = \frac{2p}{w}$ seconds per cycle and the *frequency* is then $f = \frac{1}{T} = \frac{w}{2p}$ cycles per second.

<u>Problems</u>

1) Show that $x = x(t) = c_1 \sin(wt) + c_2 \cos(wt)$ is a solution to $\frac{d^2x}{dt^2} = -w^2 x$ by direct substitution.

2) Show that $M\cos(wt + d)$ can be put into the form $c_1 \sin(wt) + c_2 \cos(wt)$ with the proper choice of constants.

3) Show that for a body obeying $x(t) = M\cos(wt + d)$, the acceleration is *directly proportional* to the displacement from equilibrium. That is, show by direct substitution that $x(t) = M\cos(wt + d)$ satisfies the differential equation of Hooke's Law: $\frac{d^2x}{dt^2} = -w^2 x$.

4) Show that a body obeying $x(t) = M\cos(wt + d)$ is at its maximum *speed* if and only if its acceleration is zero. At what positions is it when its acceleration is at a maximum?

5) Suppose a spring obeying Hooke's Law has its motion described by $x = x(t) = \frac{1}{5}\cos(8t)$.

a) Show that the motion obeys the differential equation in (1) by direct substitution. Note: by showing that $\frac{d^2x}{dt^2} = -w^2 x$ you are in fact showing that the motion is in fact *simple harmonic motion*.

b) Determine the times when the displacement from equilibrium is at a maximum and the times when the speed is maximum.

c) Interpret your results from (b) in "everyday" terms using your intuition about a mass oscillating in this manner.

Example: Suppose that the motion of a particle satisfies the following: $s''(t) = -4p^2 s(t)$, $s(0)=1$, $v(0)=2p\sqrt{3}$. Find the period, frequency and amplitude of the motion and its position, velocity and acceleration when $t=1$.

Solution: Since $s''(t) = -4p^2 s(t)$ we have that the acceleration is proportional to the position and so the particle is in *simple harmonic motion*. Therefore, it follows that $s(t) = c_1 \sin(v\,t) + c_2 \cos(v\,t)$ and since $s''(t) = -v^2 s(t)$ it follows that $-4p^2 = -v^2$ so $v = 2p$.

Therefore, $s(t) = c_1 \sin(2p\,t) + c_2 \cos(2p\,t)$. Since $s(0)=1$ we get $1 = c_1 \cdot 0 + c_2 \cdot 1$ so that $c_2 = 1$ and we can now write $s(t) = c_1 \sin(2p\,t) + \cos(2p\,t)$.

Now, $v(t) = s'(t) = 2p\,c_1 \cos(2p\,t) - 2p \sin(2p\,t)$ and we now use the fact that $v(0) = 2p\sqrt{3}$ to get $2p\,c_1 \cdot 1 + 0 = 2p\sqrt{3}$ so that $c_1 = \sqrt{3}$.

We can therefore write that $\boxed{s(t) = \sqrt{3}\sin(2p\,t) + \cos(2p\,t)}$. This is enough to give us the period and frequency but not the amplitude.

Recall:
$$\begin{aligned} x(t) &= M\cos(wt + d) = M\left[\cos v\,t \cos d - \sin v\,t \sin d\right] \\ &= M\cos d \cdot \cos v + (-M\sin d)\sin v\,t \\ &= c_1 \sin v\,t + c_2 \cos v\,t = \sqrt{3}\sin 2p\,t + \cos 2p\,t \end{aligned}$$

So, it must be that $-M\sin d = \sqrt{3}$ or $\sin d = \dfrac{-\sqrt{3}}{M}$ and $\cos d = \dfrac{1}{M}$. It follows that $\sin^2 d + \cos^2 d = 1 = \dfrac{3}{M^2} + \dfrac{1}{M^2} = \dfrac{4}{M^2}$ so that $M^2 = 4$ and $M = \pm 2$. It follows from this that the **amplitude of the motion** is 2 and this corresponds to the maximum displacement of the particle from equilibrium. The **period** is $T = \dfrac{2p}{v} = \dfrac{2p}{2p} = 1$ and the **frequency** is $f = \dfrac{1}{T} = 1$.

At time $t=1$ the position is $s(1)=1$, the velocity is $s'(1) = 2\sqrt{3}p$ and the acceleration is $s''(0) = -4p^2 \cdot 1 = -4p^2$.

6) A particle is moving along a straight line and its position at time *t* is given by $s(t) = \sin\left(\dfrac{p}{2}t\right) + \cos\left(\dfrac{p}{2}t\right)$, $t \in [0,4)$.

(a) Find the times at which the particle changes direction.
(b) Show that the particle is in *simple harmonic motion*.
(c) Find the amplitude of the motion.