$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

First, recall from class that if you invest \$1.00 for one year at the interest rate of 100% per annum (hard to find a bank that will do it) and the interest is compounded n time per year then the Balance at the end of the year is $B = \left(1 + \frac{1}{n}\right)^n$. Through tabulated values we showed that you could NEVER earn more that \$2.72 in one year if you could find such a bank. In a more advanced Calculus course (called Analysis) you can show that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$ exists and is an irrational number (and is given the symbol *e*). You can easily calculate that $e \approx 2.71828$ (and that is where the figure of \$2.72 comes from).

Fact: $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$ (G.F.G.)

We need one more

Fact: "The limit of the log is the log of the limit."

Proof: G.F.G.

Q.E.D.

Theorem. $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Proof: Let $f(x) = \ln x$.

$$f'(x) = \lim_{h\to 0^+} \frac{f(x+h) - f(x)}{h}$$ (notice: I am only doing the right hand limit here)

$$= \lim_{h\to 0^+} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0^+} \frac{\ln\left(\frac{x+h}{x}\right)}{h} = \lim_{h\to 0^+} \frac{\ln\left(1 + \frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0^+} \frac{1}{h}\ln\left(1 + \frac{h}{x}\right) = \lim_{h\to 0^+} \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \ln\left(\lim_{h\to 0^+}\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}\right)$$

Now, we do some magic. Make the substitution $n = \frac{x}{h}$ so that $\frac{h}{x} = \frac{1}{n}$ and $\frac{1}{h} = \frac{n}{x}$ and notice that for any positive value of *x* (and *x* must be positive since the domain of ln(*x*) is $\{x \mid x > 0\}$), *x* approaching zero from the right is the same as *n* approaching infinity. It follows that:

$$
\begin{aligned}
f'(x) &= \ln\left(\lim_{h\to 0^+}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right)\\
&= \ln\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{x}}\right) = \ln\left(\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^{n}\right)^{\frac{1}{x}}\right)\\
&= \ln\left(\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right)^{\frac{1}{x}}\right) = \ln\left(e^{\frac{1}{x}}\right)\\
&= \frac{1}{x}\ln e\\
&= \frac{1}{x}
\end{aligned}
$$

Q.E.D.

Of course, the above only shows that the "right hand derivative" of $\ln x$ is $\frac{1}{x}$. To show that the left hand limit is also $\frac{1}{x}$ takes a little more work. Trust me – it works. We will take it as yet another G.F.G.

Theorem. (Chain rule version) $\frac{d}{dx}\ln\left(g(x)\right) = \frac{1}{g(x)}g'(x)$.

Proof: Let $y = \ln u$ and $u = g(x)$. Then by the chain rule it follows that

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot g'(x) = \frac{1}{g(x)}\cdot g'(x).$$

Q.E.D.

Theorem. $\frac{d}{dx}e^x = e^x$

Proof: Let $y = e^x$. Then $\ln y = x$ and using *implicit differentiation* we get that

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 \text{ so that } \frac{\mathrm{dy}}{\mathrm{dx}} = y = e^x$$

Q.E.D.