<u>The Product Rule</u>

Approach 1.

Suppose that $K(x) = f(x)\cdot g(x)$ where f and g are differentiable, then $K'(x) = f'(x)g(x) + f(x)g'(x)$.

<u>Proof</u>:

$$K'(x) = \lim_{h\to 0}\frac{K(x+h)-K(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \quad \text{(by definition)}$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h)\left[f(x+h)-f(x)\right]+f(x)\left[g(x+h)-g(x)\right]}{h}$$

$$= \lim_{h\to 0}\left[g(x+h)\frac{f(x+h)-f(x)}{h}+f(x)\frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0}g(x+h)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \; {}^{1}$$

$$= \boxed{g(x)f'(x)+f(x)g'(x)}$$

Q.E.D.

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[1] Fact: If g is a **continuous** function then $\lim_{h\to 0} g\left(x+h\right) = g\left(\lim_{h\to 0}\left(x+h\right)\right) = g\left(x\right)$. This is a consequence of **Theorem 8** on page 129 of the text and is proved in Appendix F but the proof is rather technical and beyond the scope of our course. Suffice it to say that the "proof" of many of the differentiability theorems depends on this fact and we know that g **is** continuous since the hypothesis states that g is differentiable and we did prove that **differentiability implies continuity**.

Approach 2.

Suppose that $K(x) = f(x) \cdot g(x)$ where f and g are differentiable, then $K'(x) = f'(x)g(x) + f(x)g'(x)$.

Proof:

$$K'(a) = \lim_{x\to a}\frac{K(x)-K(a)}{x-a} = \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}$$

$$= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}$$

$$= \lim_{x\to a}\frac{g(x)\left[f(x)-f(a)\right]+f(a)\left[g(x)-g(a)\right]}{x-a}$$

$$= \lim_{x\to a}\left[g(x)\frac{f(x)-f(a)}{x-a}+f(a)\frac{g(x)-g(a)}{x-a}\right]$$

$$= \lim_{x\to a}g(x)\lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}f(a)\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

$$= g(a)f'(a)+f(a)g'(a)$$ [2]

Since this is true for **any** number $a \in D_K$ it follows that $\boxed{K'(x) = f'(x)g(x) + f(x)g'(x)}$.

Q.E.D.

[2] Since $g(x)$ is continuous (since it is differentiable) we know that $\lim_{x\to a} g(x) = g(a)$.