REVIEW OF FUNCTIONS – I

Definition and Domain Recall, a function is a rule that assigns a value to each element of a set (the **Domain of f**, written by me as D_f) another number in the set called the **Range of *f*** (or the **Image of *f*** and written by me as R_f). The only requirement to be a function is that the rule assigns each element of D_f to **one and only one** element of R_f.

If *f* is a function then $y = f(x)$ "says" that *f* gives to $x \in D_f$ the value $y \in R_f$. For example:

$f(x) = 17$ is the rule *f* that assigns the number 17 to every real number *x*.

$g(x) = \sqrt{x}$ is the rule *g* that assigns to every non-negative real number its principal (positive) square root. $D_g = \{x \in \mathbb{R} \mid x \geq 0\}$ and $R_f = \{x \in \mathbb{R} \mid x \geq 0\}$.

e.g. Find the domains of the following and sketch their graphs.

1) $h(x) = \sqrt{x-3}$

$D_h = \{x \mid x-3 \geq 0\} = \{x \mid x \geq 3\} = [3,\infty)$ (see page **A4** for a table of the proper interval notation for this and subsequent courses).

2) $f(x) = \dfrac{1}{x^2+4x-5}$

$D_f = \{x \mid x^2+4x-5 \neq 0\}$ and $x^2+4x-5=0 \Leftrightarrow (x+5)(x-1)=0$

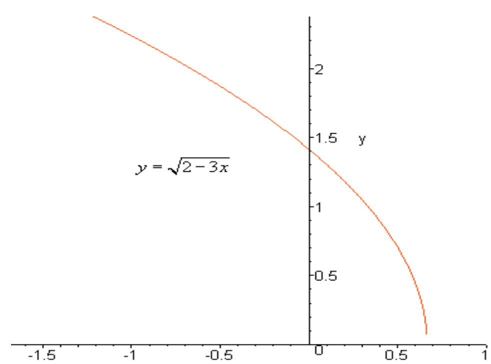
$\Leftrightarrow x=-5 \text{ or } x=1$

Therefore, $D_f = \mathbb{R} \setminus \{-5,1\}$.

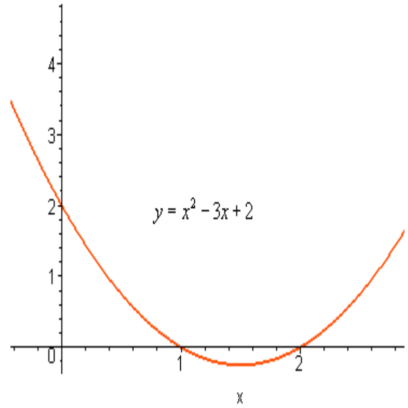
Non Linear Inequalities

Solving **linear inequalities** is easy:

e.g. Find the domain of $y = \sqrt{2-3x}$ (pictured to the right)

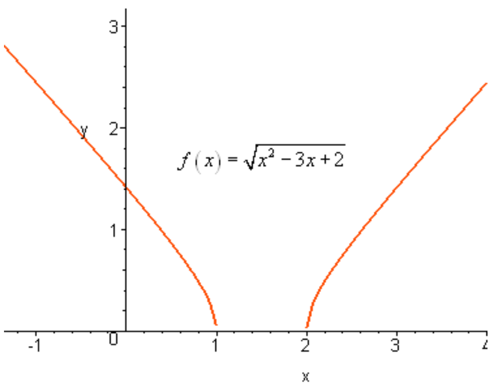


Problem 1: use the sketch of $y = x^2 - 3x + 2$ to find the domain of $f(x) = \sqrt{x^2 - 3x + 2}$.



Note: $x^2 - 3x + 2 = (x-1)(x-2)$ and so has x-intercepts at $x = 1$ and $x = 2$. From the sketch you can read that $y = x^2 - 3x + 2 \geq 0$ for $x \in (-\infty, 1] \cup [2, \infty)$ and so it follows that

$D_y = \{x \mid x^2 - 3x + 2 \geq 0\} = (-\infty, 1] \cup [2, \infty)$.



See the accompanying Maple Plot of $y = f(x) = \sqrt{x^2 - 3x + 2}$ to confirm the domain solution.

Problem 2: Use a sign table to solve $\dfrac{2x-1}{x-5} < 3$.

General Approach to Solving $f(x) > 0$ or $f(x) < 0$

1. Simplify f as much as possible (factor, combine fractions, etc…)
2. Determine the key numbers of f: the numbers where f is zero or undefined.
3. Determine the sign of $y = f(x)$ on each of the intervals determined by the key numbers by plugging in a test value.
4. Read off the answer!

Solution: $\dfrac{2x-1}{x-5} < 3 \Rightarrow \dfrac{2x-1}{x-5} - 3 < 0 \Rightarrow \dfrac{14-x}{x-5} < 0$ (Key numbers: $x = 5,\ x = 14$)

x	**test value**	**sign of** $(14-x)/(x-5)$	**Conclusion**
$(-\infty, 5)$	0	**neg**	$f(x) < 0$
$(5, 14)$	10	**pos**	$f(x) > 0$
$(14, \infty)$	20	**neg**	$f(x) < 0$

Clearly, $\dfrac{2x-1}{x-5} < 3 \Leftrightarrow \dfrac{14-x}{x-5} < 0 \Leftrightarrow x \in (-\infty, 5) \cup (14, \infty)$

Piece-wise defined functions

Problem 3: Plot the following and state their domains and ranges:

1) $g(x)=\begin{cases} -2 & x\le -2 \\ 1-x^2 & -2<x<1 \\ -x & x\ge 1 \end{cases}$ 2) $y=|x-3|$

Vertical and Horizontal Shifts If $k>0$ then the graph of:

$y=f(x)+k$ is the graph of $y=f(x)$ shifted *k* units **up**
$y=f(x)-k$ is the graph of $y=f(x)$ shifted *k* units **down**
$y=f(x+k)$ is the graph of $y=f(x)$ shifted *k* units to the **left**
$y=f(x-k)$ is the graph of $y=f(x)$ shifted *k* units to the **right**.

Problem 4: Use the graph of $y=\sqrt{x}$ to sketch the graphs of

$y=\sqrt{x}+3$ 2) $y=\sqrt{x}-2$ 3) $y=\sqrt{x+1}$ 4) $y=\sqrt{x-2}$

Composition: The **composition** of f and g is $(f\circ g)(x)=f(g(x))$

Problem 5: Let $f(x)=1-x+x^2$ and $g(x)=\sqrt{x}$ and show that:

$f(g(x))=1-\sqrt{x}+x$ $g(f(x))=\sqrt{1-x+x^2}$

$\frac{f(1+h)-f(1)}{h}=1+h$ $\frac{g(x+h)-g(x)}{h}=\frac{1}{\sqrt{x+h}+\sqrt{x}}$

Linear Functions: The equation of the straight line with slope *m* and passing through the point (x_0,y_0) is given by:

$y-y_0=m(x-x_0)$ (point-slope form)

The equation of the straight line with slope *m* and *y*-intercept *b* is given by:

$y=mx+b$ (slope-intercept form)

Two lines, $y=m_1x+b_1$ and $y=m_2x+b_2$, are:

parallel iff $m_1=m_2$

perpendicular iff $m_1=\frac{-1}{m_2}$ (or $m_1m_2=-1$)

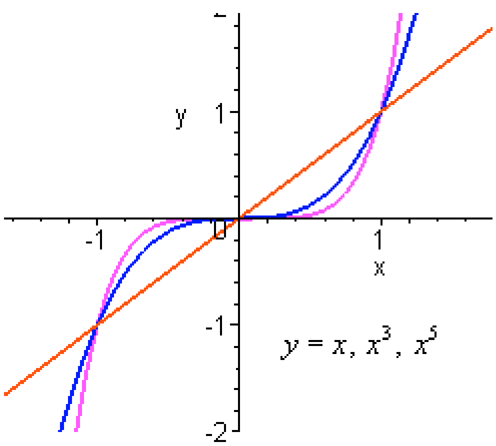
- $y = k$ (literally $y = 0 \cdot x + k$) is a horizontal line with *y*-intercept k (and slope $m = 0$)
- $x = k$ is a vertical line with *x*-intercept k (and **undefined** slope)

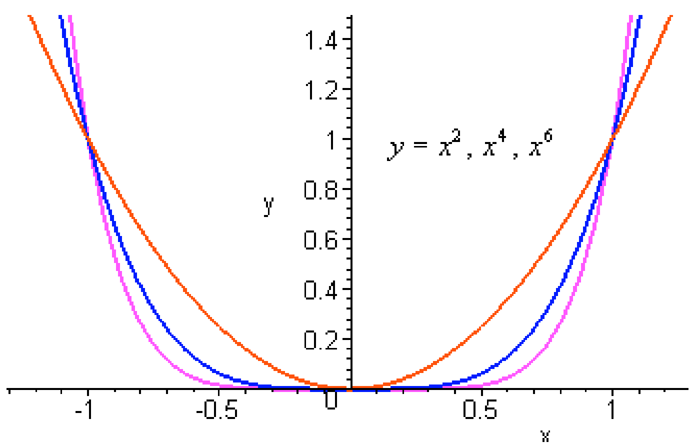
Quadratic Functions: The graph of $f(x) = ax^2 + bx + c$ is a parabola

- with vertex at $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$
- with *x*-intercepts at $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- opening upward if $a > 0$
- opening downward if $a < 0$

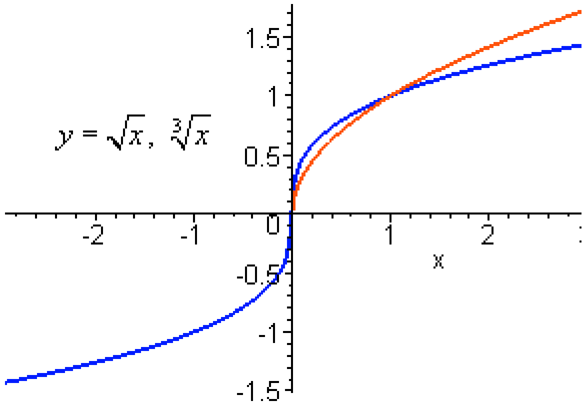
<u>Problem 6</u>: Shade the region bounded by $y = x$ and $y = 2 - x - x^2$ and find their intersection.

Power Functions:





Root functions:



Inverse Functions If f is a 1-1 function then f has an inverse, denoted f^{-1}, defined by

$$\boxed{f(a)=b \Leftrightarrow f^{-1}(b)=a} \text{ for all } a\in D_f \text{ and } b\in R_f = D_{f^{-1}}$$

This definition is equivalent to the **cancellation equations**:

$$f\left(f^{-1}(b)\right)=b \text{ and } f^{-1}\left(f(a)=a\right) \text{ for all } a\in D_f \text{ and } b\in R_f = D_{f^{-1}}$$

Note: The job of the **inverse function** f^{-1} is to undo whatever work the function f does. Therefore, it follows that if f is a function with domain D_f and range R_f then the **domain of** f^{-1} is the range of f ($D_{f^{-1}} = R_f$) and the **range of** f^{-1} is the domain of f ($R_{f^{-1}} = D_f$).

Problem 7: Use the fact that if $y = f^{-1}(x)$ then $x = f(y)$ to find f^{-1} if $f(x)=1-2x$ and show that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right)$.

Solution

$$y = f^{-1}(x)$$
$$f(y)=x$$
$$1-2y=x$$
$$y=\frac{1-x}{2} \qquad \text{Therefore, } f^{-1}(x)=\frac{1-x}{2}$$

$$f\left(f^{-1}(x)\right)=1-2\left(f^{-1}(x)\right)=1-2\cdot\frac{1-x}{2}=1-(1-x)=x$$

$$f^{-1}\left(f(x)\right)=\frac{1-f(x)}{2}=\frac{1-(1-2x)}{2}=\frac{2x}{2}=x$$