The Average Value of a Function

Suppose that $y = f(x)$ is a continuous function on [a,b]. To get an <u>estimate</u> of the average value of *f* on [a,b] we divide the interval into *n* equal subintervals by numbering the x-axis as follows: $a = x_0,\ x_1, x_2, x_3,\ \dots, x_i, \dots, x_n = b$ where $x_i = a + i\Delta x$ and $\Delta x = \dfrac{b-a}{n}$. Now we get that the average value of *y* is:

$$\begin{aligned} y_{AVE} &\approx \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \\ &\approx \sum_{i=1}^{n} f(x_i)\frac{1}{n} \\ &\approx \sum_{i=1}^{n} f(x_i)\frac{1}{\dfrac{b-a}{\Delta x}} \quad \text{since } \Delta x = \frac{b-a}{n} \\ &\approx \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x \end{aligned}$$

It seems natural to define <u>the</u> average value of *f* to be $\lim_{n\to\infty} y_{AVE}$:

$$\boxed{y_{AVE} = \lim_{n\to\infty}\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx}$$

This gives us:

<u>Theorem</u>: If *f* is continuous on [a,b] then the average value of *f* on the interval [a,b] is given by $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$.

<u>Example</u>: Suppose that the temperature in a city (in degrees Celsius) t hours after 9 A.M. is modelled by $T(t) = 10 + 8\sin\left(\dfrac{pt}{12}\right)$. Find the average temperature in the city for the 12 hour period from 9 A.M. to 9 P.M.

<u>Solution</u>: $T_{ave} = \dfrac{1}{12}\displaystyle\int_0^{12}\left(10 + 8\sin\left(\frac{pt}{12}\right)\right)dt$. Usiing the substitution $u = \dfrac{pt}{12}$ we get that $T_{ave} = \dfrac{1}{p}\displaystyle\int_{u=0}^{u=p}(10 + 8\sin u)\,du = \frac{10p + 16}{p} \approx 15.1$ (Make

sure that **<u>you</u>** get the same answer!) It follows that the average temperature in the city is *approximately* 15 degrees C.