THE FUNDAMENTAL THEOREM OF CALCULUS

Definition: F is an **antiderivative** of *f* if $F'(x) = f(x)$.

Big Fact: If F and G are two antiderivatives of *f* then $G(x) = F(x) + C$ where C is an arbitrary constant. That is, antiderivatives of a function are *almost* unique.

Definition: If $F'(x) = f(x)$ then **the most general antiderivative** of *f* is $\int f(x)dx = F(x) + C$ where C is an arbitrary constant (often called the "constant of integration.") Note: depending on context, the symbol $\int f(x)dx$ may stand for *any* antiderivative of *f* or for the most general antiderivative of *f*. In our course, unless otherwise stated, it will be the most general antiderivative.

F.T.C. part I: If *f* is continuous on [a,b] and $G(x) = \int_a^x f(t)dt$ where $a \le x \le b$ then G is differentiable and $G'(x) = f(x)$. (When *f* is a positive function, G can be thought of as the "area *so far* function.")

Comment: One thing that F.T.C. part I says is that if *f* is continuous then *f* **always** has an antiderivative, namely $G(x)$!

Proof of F.T.C. part I:

G.F.G. – 'nuff said.

F.T.C. part II: If *f* is continuous on [a,b] and F is *any* antiderivative of *f* then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof (the Good News is that you are responsible for this one!)

Suppose that F is any antiderivative of *f* and that $G(x) = \int_a^x f(t)dt$. By the F.T.C. part I, G is also an antiderivative of *f*. It follows (by the Big Fact) that $G(x) = F(x) + C$ where C is an arbitrary constant.

Now, by "plugging in" we have that: $G(a) = F(a) + C$.

But, $G(a) = \int_a^a f(x)dx = 0$

and so it follows that $C = -F(a)$

so $G(x) = F(x) - F(a)$.

Now, by "plugging in" we also have $G(b) = F(b) + C$

and since $C = -F(a)$ and $G(b) = \int_a^b f(x)dx$ it follows that

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}\ .$$

Q.E.D.

<u>Notation</u>: If F is an antiderivative of f then $\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$

<u>Example</u> of **proper use** of notation:

$$\int_1^2 \left(x^2 - 2x + 1\right)dx$$

$$= \left(\frac{x^3}{3} - x^2 + x\right)\Bigg]_1^2$$

$$= \left(\frac{2^3}{3} - 2^2 + 2\right) - \left(\frac{1^3}{3} - 1^2 + 1\right) = \frac{1}{3}$$

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