Geometric Series and Annuities

Definition: A **geometric series** is a sum of the form $a+ar+ar^2+ar^3+\cdots+ar^{n-1}$ where *a* and *r* are fixed constants. The constant *r* is called the **constant ratio** and is the ration of any two successive terms.

Let $S_n = a+ar+ar^2+ar^3+\cdots+ar^{n-1}$. Then multiplying S_n by *r* we obtain $rS_n = ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n$. If we then subtract we obtain:

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n) \text{ so that}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Example 1. To determine the value of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{128}$ you must first recognize it as a geometric series of 8 terms (to help see the number of terms you can write the series in the form $1+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^7}$) with first term 1 and common ration of $r = \frac{1}{2}$. It follows that its value is $S_8 = \frac{1\left(1-\left(\frac{1}{2}\right)^8\right)}{1-\frac{1}{2}} = \frac{1-\frac{1}{256}}{\frac{1}{2}} = 2\cdot\frac{255}{256} = \frac{255}{128}$.

This can be checked by direct calculation.

Definition: An **annuity** is a series of equal payments made at the ends of equal intervals of regular time periods for a fixed number of periods. The intervals are of length equal to the available compounding period.

Example 2. Suppose your bank is offering an account that pays 5% per annum compounded quarterly (4 times/year) and you decide to deposit \$100 at the end of each period for 3 years (for a total of 12 payments). You will make your first deposit on December 31st , 2002. How much money will you have at the end of the 3rd year (i.e. at the end of the 12th period)? (Note: banks calculate the interest on a period based on the *smallest amount* present during the entire period. Therefore, make your first deposit of \$100 on December 31 so that it is present in the account for January 1 – the beginning of the first period.)

Solution. The final amount of the annuity (called the **Future Value** of the annuity will be the sum of the Future Values of each deposit. Your first deposit of \$100 will be in the account for $4\cdot 3 = 12$ full periods and hence will have a future value of $100(1+i)^{12}$ where $i = \frac{.05}{4}$ is the interest for one period. Your second deposit will be in the account for 11 periods and its future value will be $100(1+i)^{11}$. Your third deposit will be in the account for 10 periods and its future value will be $100(1+i)^{10}$ etcetera… Your last deposit will be in the account for 1 period and its future value will be $100(1+i)^{1}$. It follows that the final amount of the annuity will be the sum of all these future values:

$$F.V.=100(1+i)+100(1+i)^{2}+100(1+i)^{3}+\cdots+100(1+i)^{12}$$

The F.V. (future value) is clearly a **geometric series** with $a=100(1+i)$, common ratio $r=1+i$ and number of terms $n=12$ and so we can use the formula for the sum of n terms of a geometric series $S_n=\dfrac{a(1-r^n)}{1-r}$:

$$F.V.=\frac{100(1+i)\left[1-(1+i)^{12}\right]}{1-(1+i)}\approx \$1302.11$$

<u>Example 3</u>. Find the **present value** of an annuity that will pay \$100 each quarter for 12 payments (i.e. for 3 years). Assume an interest rate of 5% compounding quarterly. This is, determine how much money you must invest now at 5% compounding quarterly to produce an annuity that will pay \$100 for 3 years (at the end of which there will be no money in the account).

<u>Solution</u>. What you must do, in effect, is determine the present value of each \$100 payment. The present value of the annuity will then be the total of the individual present values. Recall, the present value of \$B is the principal \$P that must be invested today to produce the \$B at the end of the investment and is obtained by solving $B=P\left(1+\frac{r}{n}\right)^{nt}$ for P to obtain $PV=B\left(1+\frac{r}{n}\right)^{-nt}$. Now, *nt* is the total number of periods of the annuity and $\frac{r}{n}$ is the interest rate *i* per period. So, if the total number of periods is, say, *k*, then the formula becomes $PV=B(1+i)^{-k}$.

The present value of the first \$100 payment is $100(1+i)^{-1}$

The present value of the second \$100 payment is $100(1+i)^{-2}$

The present value of the third \$100 payment is $100(1+i)^{-3}$ and so on until

The present value of the last \$100 payment is $100(1+i)^{-12}$. It follows that the total of all these individual present values is the present value of the annuity:

$$PV = 100(1+i)^{-1} + 100(1+i)^{-2} + 100(1+i)^{-3} + \cdots + 100(1+i)^{-12}$$

$$= 100\left(\frac{1}{1+i}\right) + 100\left(\frac{1}{1+i}\right)^2 + 100\left(\frac{1}{1+i}\right)^3 + \cdots + 100\left(\frac{1}{1+i}\right)^{12}$$

$$= \frac{100\left(\frac{1}{1+i}\right)\left[1-\left(\frac{1}{1+i}\right)^{12}\right]}{1-\left(\frac{1}{1+i}\right)}$$

$$= \$1{,}107.93$$

(where I have used $i = \frac{0.05}{4}$, common ratio $r = (1+i)^{-1} = \frac{1}{1+i}$ and $n = 12$.

<u>Significance</u>: to produce a payment of \$100 every 3 months for three years you could deposit \$1,107.93 today in an account paying 5% and compounding quarterly. After the last \$100 payment the account will hold \$0.

Notice, the value of this annuity to a recipient unadjusted by interest rate is simply $\$100 \cdot 12 = \$1{,}200$ and so we have the relationship:

$$\boxed{\textit{present value} < \textit{actual value} < \textit{future value}}.$$