Integrating Trigonometric Functions — Basic

The following identities should be known by heart – they are crucial to finding antiderivatives.

- $\sin^2\theta = 1-\cos^2\theta$
- $\cos^2\theta = 1-\sin^2\theta$
- $\tan^2\theta = \sec^2\theta - 1$
- $\sec^2\theta = \tan^2\theta + 1$
- $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$
- $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

The last two both come from the identities $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$.

The next two antiderivative formulae are basic and should be memorized or their methods of derivation memorized.

- $\int \tan\theta d\theta = \ln\left|\sec\theta\right| + C$ (derivation: use $u = \cos\theta$ on $\int \dfrac{\sin\theta}{\cos\theta} d\theta$)
- $\int \sec\theta d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$ (derivation: use $u = \sec\theta + \tan\theta$ on $\int \sec\theta \cdot \dfrac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} d\theta$)

The next two antiderivative formulae are basic and follow direct from Cal I derivative formulae or, if you don't see that, they can be obtained quickly by an easy substitution. You can save yourself many steps if you "keep them in mind."

- $\int \sin(a\theta)\, d\theta = -\dfrac{1}{a}\cos(a\theta) + C$
- $\int \cos(a\theta)\, d\theta = \dfrac{1}{a}\sin(a\theta) + C$

Following are the antiderivatives of the first four powers of $\sin x$, you should make sure you can do the same work for the powers of $\cos x$.

1. $\int \sin x dx = -\cos x + C$ (on your basic list from Cal I)

2. $\int \sin^2 x dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\cdot\left[\int dx - \int \cos 2x dx\right] = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

3. $\int \sin^3 x dx = \int \sin x \cdot \sin^2 x dx = \int \sin x\left(1-\cos^2 x\right) dx$

$= \int \left(1-u^2\right)\left(-du\right) = \int \left(u^2 - 1\right) du$ using $u = \cos x$

$\frac{u^3}{3} - u + C = \frac{\cos^3 x}{3} - \cos x + C$

Therefore, $\int \sin^3 x dx = \frac{\cos^3 x}{3} - \cos x + C$

4. $\int \sin^4 x dx = \int \left(\sin^2 x\right)^2 dx = \int \left[\frac{1-\cos 2x}{2}\right]^2 dx = \int \left[\frac{1-2\cos 2x + \cos^2 2x}{4}\right] dx$

$$= \frac{1}{4}\left[\int dx - 2\int \cos 2x dx + \int \frac{1+\cos 4x}{2} dx\right]$$

$$= \frac{1}{4}\left[x - 2\cdot\frac{1}{2}\sin 2x + \frac{1}{2}\int (1+\cos 4x) dx\right]$$

$$= \frac{1}{4}\left[x - \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right] + C$$

$$= \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$$

Therefore, $\int \sin^4 x dx = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$.

An important class of antiderivatives (mainly because they arise in intermediary steps inside other calculations) are those of the form $\int \sin^m x \cos^n x dx$ where the *n* and the *m* are natural numbers. When one or the other of the *n* or *m* is odd then these integrals are "easy."

5. $\int \sin^m x \cos^n x dx$ when *m* is odd.

Note that when *m* is odd it follows that *m* can be written in the form $m = 2k+1$ for some natural number *k*. In this case we can proceed as follows:

$$\int \sin^{2k+1} x \cos^n x dx = \int \sin^{2k} x \cos^n x \sin x dx = \int \left(\sin^2 x\right)^k \cos^n x \sin x dx$$

$$= \int \left(1-\cos^2 x\right)^k \cos^n x \sin x dx = \int \left(1-u^2\right)^k u^n \left(-du\right) \text{ where } u = \cos x$$

$= -\int \left(1-u^2\right)u^n du$ and then you can proceed using only elementary tools.

(If you look back at integral #3 above, you should be able to recognize it as an example of this form with the power of sine odd and the cosine term of power zero.)

Example:

$$\int \sin^5 x \cos^4 x dx = \int \sin^4 x \cos^4 x \sin x dx = \int \left(1-\cos^2 x\right)^2 \cos^4 x \sin x dx$$

$$= \int \left(1-u^2\right)^2 u^4 \left(-du\right) = \int \left(-u^8 + 2u^6 - u^4\right) du = -\frac{u^9}{9} + 2\frac{u^7}{7} - \frac{u^5}{5} + C$$

$= -\frac{\cos^9 x}{9} + \frac{2\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$. (where I have used the substitution $u = \cos x;\ du = -\sin x dx$)

6. $\int \sin^m x \cos^n x dx$ when *n* is odd.

Same approach as in #5 except that you "split off" a factor of $\cos x$ to go with the dx and convert the remaining even power of cosine to sines by using $\cos^2\theta = 1-\sin^2\theta$. You then let $u = \sin x$ and $du = \cos x dx$.

7. $\int \sin^m x \cos^n x dx$ when *both m* and *m* are odd.

In this case you can proceed as in #5 or #7, being aware that your answer will "look" different according to which approach you chose.

8. $\int \sin^m x \cos^n x dx$ when *both m* and *n* are even.

This is the "worst case scenario" in terms of the amount of work involved. In this unfortunate case you must convert the integrand completely into an even power of either sine or cosine and then proceed as in #2 or #4.

Example:

$$\int \sin^2 x \cos^2 x dx = \int \sin^2 x\left(1-\sin^2 x\right)dx$$

$$= \int\left(\sin^2 x - \sin^4 x\right)dx = \int \frac{1-\cos 2x}{2}dx - \int\left(\frac{1-\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) - \frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) - \frac{1}{4}\left(x - \sin 2x + \int\frac{1+\cos 4x}{2}dx\right)$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} - \frac{x}{4} + \frac{\sin 2x}{4} - \frac{1}{8}\left(x + \frac{1}{4}\sin 4x\right) + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

9. $\int \sec x dx$ -- requires substitution:

$$\int \sec x dx = \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

let $u = \sec x + \tan x$ so that $du = \sec^2 x + \sec x \tan x$ and then

$$\int \sec x dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$$

10. $\int \sec^2 x dx = \tan x + C$ -- basic list!

11. $\int \sec^3 x dx$ -- hard! Requires integration by parts and some ingenuity! You should be able to handle it though – see below.

Let $I = \int \sec^3 x dx$ and then let
$$\boxed{\begin{array}{ll} u = \sec x & dv = \sec^2 x dx \\ du = \sec x \tan x dx & v = \tan x \end{array}}$$

$$\Rightarrow I = uv - \int v du = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$= \sec x \tan x - \int \sec x\left(\sec^2 x - 1\right)dx$$

$$= \sec x \tan x - \left[\int \sec^3 x dx - \int \sec x dx\right]$$

$$= \sec x \tan x - I + \ln\left|\sec x + \tan x\right|$$ ($\int sec^3 x dx = I$, the original integral)

$$2I = \sec x \tan x + \ln\left|\sec x + \tan x\right|$$

$$I = \frac{1}{2}\left[\sec x + \tan x + \ln\left|\sec x + \tan x\right|\right] + C$$

12. $\int \tan^m x \sec^n x dx$ when *n* is even.

The approach in this situation is to put a factor of $\sec^2 x$ with the *dx* and then convert the remaining **even power of secant** into a function of tangent using the identity $\sec^2 x = \tan^2 + 1$ and the let $u = \tan x$

Example:

$$\int \tan^4 x \sec^4 x dx = \int \tan^4 x \sec^2 x \cdot \sec^2 x dx$$

$$= \int \tan^4 x \left(\tan^2 x + 1\right) \sec^2 x dx \text{ ; let } u = \tan x \text{ ; } du = \sec^2 x dx$$

$$= \int u^4 \left(u^2 + 1\right) du = \int \left(u^6 + u^4\right) du = \frac{u^7}{7} + \frac{u^5}{5} + C$$

$$= \frac{\tan^7 x}{7} + \frac{\tan^5 x}{5} + C$$

13. $\int \tan^m x \sec^n x dx$ when *m* is odd.

The approach here is to take a factor of $\sec x \tan x$ to put with the *dx* and rewrite the remaining even power of tangent in terms of secant using $\tan^2 x = \sec^2 x - 1$. You can then use the substitution $u = \sec x$.

Example: $$\int_0^{\pi/4} \tan^3 x \sec^4 x dx = \int_0^{\pi/4} \tan^2 x \sec^3 x \sec x \tan x dx$$

$$= \int_0^{\pi/4} \left(\sec^2 x - 1\right) \sec^3 x \sec x \tan x dx$$

Now let $u = \sec x, \; du = \sec x \tan x dx$ and note that:

$$x = 0 \Rightarrow u = \sec(0) = 1; \; x = \frac{\pi}{4} \Rightarrow u = \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\pi/4\right)} = \frac{1}{1/\sqrt{2}} = \sqrt{2} \text{ so that}$$

$$\int_0^{\pi/4} \tan^3 x \sec^4 x dx = \int_1^{\sqrt{2}} \left(u^2 - 1\right) u^3 du = \int_1^{\sqrt{2}} \left(u^5 - u^3\right) du$$

$$= \left(\frac{u^6}{6} - \frac{u^4}{4}\right)\Bigg]_1^{\sqrt{2}} = \left(\frac{\left(\sqrt{2}\right)^6}{6} - \frac{\left(\sqrt{2}\right)^4}{4}\right) - \left(\frac{1}{6} - \frac{1}{4}\right) = \frac{5}{12}$$