The Present Value of Future Money

Recall: The **Present Value** of the future (*t* years hence) money \$B is:

i) $P = B\left(1+\frac{r}{n}\right)^{-nt}$ if money is compounded *n* times per year for *t* years at the interest rate *r*.

ii) $P = Be^{-rt}$ if the money is to be compounded continuously for *t* years at the interest rate *r*.

The present value of \$B of future money *t* years from now represents the amount of money you would have to invest today at the prevailing interest rate in order to produce the \$B in *t* years.

Example 1. Suppose you want to have \$10,000 available 10 years hence and that you have available an investment vehicle that will pay 4% interest on (a) a quarterly basis and (b) on a monthly basis. Then, the Present Value of the \$10,000 in each case is:

a) $PV = 10000\left(1+\frac{.04}{4}\right)^{-4\cdot 10} = 10000\left(\frac{4.04}{4}\right)^{-40} = \$6,716.53$

b) $PV = 10000\left(1+\frac{.04}{12}\right)^{-12\cdot 10} = 10000\left(\frac{12.04}{12}\right)^{-120} = \$6,707.66$

Example 2. Suppose you want to have \$10,000 available 10 years hence and that you have available an investment vehicle that will pay 4% interest compounding continuously. Then, the Present Value of the \$10,000 is:

$$PV = 10000e^{-(0.04)(10)} = 10000e^{-0.4} = \$6,703.20$$

Notice: as the number of compounding periods increases, the better the continuous case approximates the discrete case.

Present Value of an Income Stream Suppose that money is being generated in a continuous fashion (i.e. an **income stream**) at the rate of $\$f(t)/year$ over a period of *T* years and being invested (as it is being generated) at the rate of *r* per annum compounding continuously. What would be the Present Value of such an income stream? To calculate this we would divide the time interval [0,*T*] into *i* equal subintervals and consider the amount generated by the income stream on the i^{th} interval. The amount generated will be $B_i \approx f(t_i)\Delta t$ and its present value would be $PV_i \approx f\left(t_i\right)\Delta t e^{-rt_i} \approx f\left(t_i\right)e^{-rt_i}\Delta t$. To approximate the present value of the entire stream then we use $PV \approx \sum_{i=1}^{n} f(t_i)e^{-rt_i}\Delta t$ and it quickly follows that

$$PV = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)e^{-rt_i}\Delta t = \int_0^T f(t)e^{-rt}dt$$

Example 3 Suppose that a 5 year franchise for an ice cream parlour is projected to generate profit at the rate $f(t) = 14000 + 490t$ dollars per year. If the resulting income stream was to be invested at the rate of 4% per annum compounding continuously then its present value would be

$$PV = \int_0^5 (14000 + 490t)e^{-0.04t}dt = \$68{,}810.68$$

You should confirm the above result using *integration by parts* with $u = 14000 + 490t$ and $dV = e^{-.04t}$. If you cannot, then bring your work to my office so we can together figure out what is wrong.

Example 4 Suppose that you just want to determine the actual profit (unadjusted by the investment) that the franchise will generate if the income is simply put in your pocket as it comes in. To calculate this we simply take the limit of the Reimann Sum that adds up the profits generated on each time subinterval. This gives:

$$\text{Profit} = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta t = \int_0^T f(t)dt\,.$$

So, for the ice cream parlour we see that:

$$\text{Profit} = \int_0^5 (14000 + 490t)\,dt = \$76{,}125$$

Recall that the **future value** of the parlour is given by:

$$\begin{aligned} FV &= \int_0^T f(t)e^{r(T-t)}dt \\ &= \int_0^5 (14000 + 490t)e^{0.04(5-t)}dt = \$84{,}045.56 \end{aligned}$$

Notice that **Present Value < Actual Value < Future Value**. This, and you should be able to see this if you think about it, will always be true.