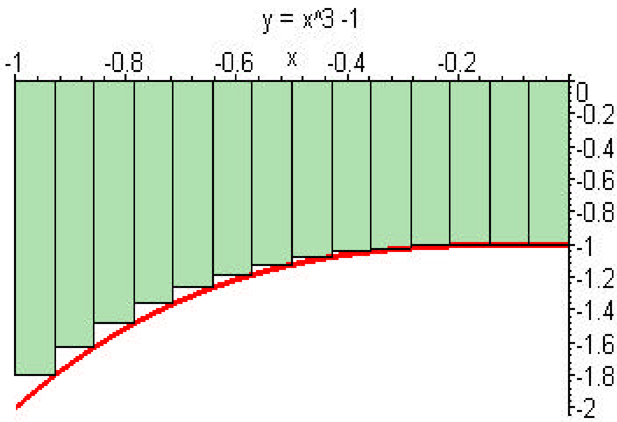
A Riemann Sum Calculation

Recall: $\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$ where x_i is the right endpoint of the ith approximating rectangle and Δx is the width of the base of each rectangle. For *every* problem, $\Delta x = \dfrac{\text{length of interval}}{n} = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$.

Terminology: In the definition the $\sum_{i=1}^{n} f(x_i)\Delta x$ part is called a **Riemann sum** to honour of the great German mathematician Bernhard Riemann.

Example: Use the above definition to find $\int_{-1}^{0}(x^3-1)dx$.



See the accompanying sketch which shows the region in question with 14 approximating rectangles. The first task is to find the product $f(x_i)\Delta x$ for the ith approximating rectangle.

$\Delta x = \dfrac{0-(-1)}{n} = \dfrac{1}{n}$ and

$x_i = -1 + i\Delta x = -1 + \dfrac{i}{n}$ so it follows that

$$f(x_i)\Delta x = f\left(-1+\frac{i}{n}\right)\cdot\frac{1}{n}$$

$$=\left[\left(-1+\frac{i}{n}\right)^3 - 1\right]\cdot\frac{1}{n}$$

$$=\left[-2+\frac{3i}{n}-\frac{3i^2}{n^2}+\frac{i^3}{n^3}\right]\cdot\frac{1}{n}$$

$$=-\frac{2}{n}+\frac{3i}{n^2}-\frac{3i^2}{n^3}+\frac{i^3}{n^4}$$

Therefore,

$$\sum_{i=1}^{n} f\left(x_i\right)\Delta x = \sum_{i=1}^{n}\left(-\frac{2}{n}+\frac{3i}{n^2}-\frac{3i^2}{n^3}+\frac{i^3}{n^4}\right)$$

$$=-\frac{2}{n}\sum_{i=1}^{n}1+\frac{3}{n^2}\sum_{i=1}^{n}i-\frac{3}{n^3}\sum_{i=1}^{n}i^2+\frac{1}{n^4}\sum_{i=1}^{n}i^3$$

$$=\frac{-2}{n}n+\frac{3}{n^2}\frac{n(n+1)}{2}-\frac{3}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{1}{n^4}\left(\frac{n(n+1)}{n}\right)^2$$

$$=-2+\frac{3n+3}{2n}-\frac{2n^2+3n+1}{2n^2}+\frac{n^2+2n+1}{4n^2}$$

Note: you should **verify** the algebra steps used to produce the simplification above.

The hard work is done. All that remains now is to find the limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(x_i\right)\Delta x = \lim_{n\to\infty}\left(-2+\frac{3n+3}{2n}-\frac{2n^2+3n+1}{2n^2}+\frac{n^2+2n+1}{4n^2}\right)$$

$$=-2+\frac{3}{2}-\frac{2}{2}+\frac{1}{4}=-\frac{5}{4}$$

(Note, the limits are found exactly as you found them in your Cal I course.)

It follows that $\int_{-1}^{0}\left(x^3-1\right)dx=-\frac{5}{4}$. The result is negative because the graph is **below** the x-axis (i.e. *f* is a negative function), it represents the negative of the area above the graph of $y = x^3 - 1$ and below the interval [-1,0].