

## The Sum of an Infinite Geometric Series

**Big Fact:** If  $r$  is a constant such that  $-1 < r < 1$  or, equivalently,  $|r| < 1$  then

$$\lim_{n \rightarrow \infty} r^n = 0. \text{ Proof: G.F.G.}$$

**Consequence:** If  $a$  is a constant and  $r$  is a constant,  $-1 < r < 1$ , then

$$\lim_{n \rightarrow \infty} \frac{a(1-r^n)}{1-r} = a \frac{\lim_{n \rightarrow \infty} (1-r^n)}{\lim_{n \rightarrow \infty} (1-r)} = a \frac{1-0}{1-r} = \frac{a}{1-r}.$$

Now, let  $S_\infty = a + ar + ar^2 + ar^3 + \dots = \sum_{i=1}^{\infty} ar^{i-1}$ . By the sum of the **infinite series**  $S_\infty$  we mean  $\lim_{n \rightarrow \infty} S_n$  if the limit exists. By the BIG FACT we know that if  $-1 < r < 1$  then the limit exists and we get that  $S_\infty = \lim_{n \rightarrow \infty} S_n = \frac{a}{1-r}$  where  $a$  is the first term of the series and  $r$  is the constant ratio.

**Example 1**  $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$  is an infinite geometric series with  $a = \frac{1}{2}$  and  $r = \frac{1}{2}$

$$\text{and so it follows that } S_\infty = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

**Example 2**  $1 - \frac{1}{3} + \frac{1}{27} - \frac{1}{81} + \dots$  is a geometric series with  $a = 1$  and  $r = -\frac{1}{3}$  and so

$$S_\infty = \frac{1}{1 - \left(-\frac{1}{3}\right)} = \frac{3}{4}.$$

**Example 3** Express  $0.\bar{4} = 0.4444\dots$  as a fraction in reduced form.

**Solution:**

$$0.4444\dots = .4 + .04 + .004 + .0004 + \dots$$

$$= \frac{4}{10} + \frac{4}{100} + \frac{4}{1000} + \frac{4}{10000} + \dots$$

**Notice:**

So 0.444... is really an infinite geometric series with  $a = \frac{4}{10}$  and  $r = \frac{1}{10}$  and so we

$$\text{get that } 0.444\dots = S_{\infty} = \frac{\frac{4}{10}}{1 - \frac{1}{10}} = \frac{\frac{4}{10}}{\frac{9}{10}} = \frac{4}{9}$$

Example 2 Find the reduced fraction that corresponds to 1.232323....

Solution  $1.2323\dots = 1 + \frac{23}{100} + \frac{23}{10000} + \frac{23}{100000} + \dots$  and you can see that the terms

after the one on the right hand side form a geometric series with  $a = \frac{23}{100}$  and  $r = \frac{1}{100}$  and

$$\text{so we get } 1.2323\dots = 1 + \frac{\frac{23}{100}}{1 - \frac{1}{100}} = 1 + \frac{\frac{23}{100}}{\frac{99}{100}} = 1 + \frac{23}{99} = \frac{122}{99}.$$