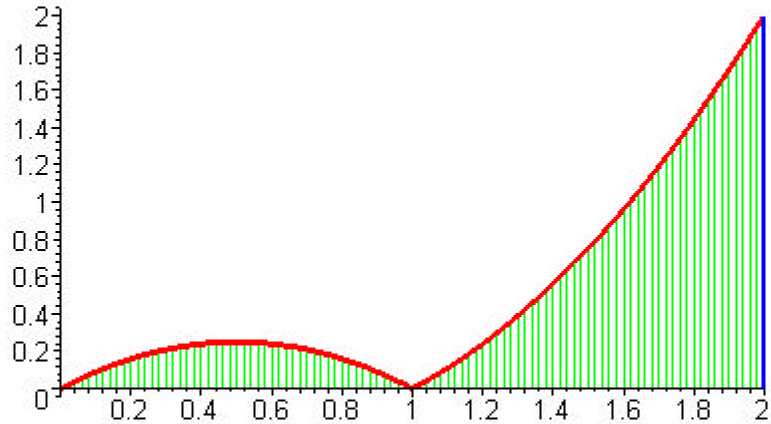
An Area Problem

Consider the area bounded by $f(x) = |x - x^2|$ for $x \in [0,2]$. A quick sketch of the familiar $y = x - x^2$ should convince you that *f* can be written as $f(x) = \begin{cases} f_1(x) & x \in [0,1] \\ f_2(x) & x \in (1,2] \end{cases}$ where $f_1(x) = x - x^2$ and $f_2(x) = -(x - x^2)$. A sketch of the region in question is given below.



It follows that the area in question is given by

$$A = \int_0^1 f_1(x)dx + \int_1^2 f_2(x)dx = \frac{1}{6} + \frac{5}{6} = 1 \ unit^2$$

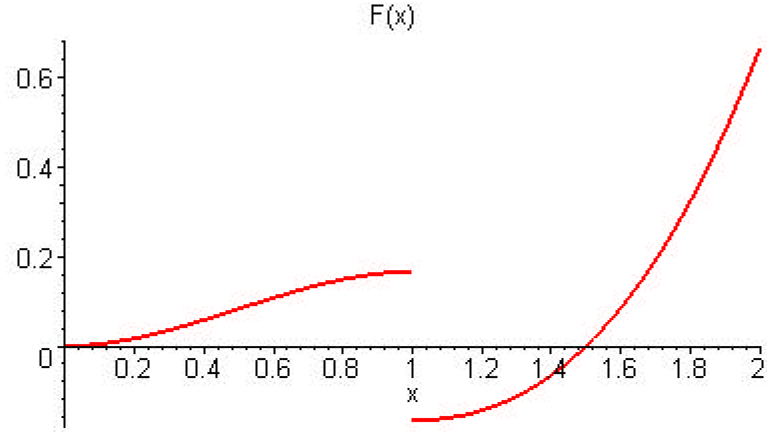
(You should verify the numbers.)

Some clever folk may want to use the *Fundamental Theorem of Calculus* directly on the function *f* as follows.

Let $F(x) = \begin{cases} \dfrac{x^2}{2} - \dfrac{x^3}{3} & x \in [0,1] \\ -\dfrac{x^2}{2} + \dfrac{x^3}{3} & x \in (1,2] \end{cases}$ which looks like an antiderivative of *f* on the interval [0,2]. Since the function *f* is continuous on [0,2] . The F.T.C. is very clear when it states that if *f* is continuous on [a,b] (which our *f* is) then $\int_a^b f(x)dx = F(b) - F(a)$ where *F* is **any** antiderivative of *f* on [a,b]. Reasoning this way it should be that the area in question is $A = \int_0^2 |x - x^2| dx = F(2) - F(0) = \frac{2}{3} units^2$ which is **NOT** correct (see above!).

What has gone wrong??

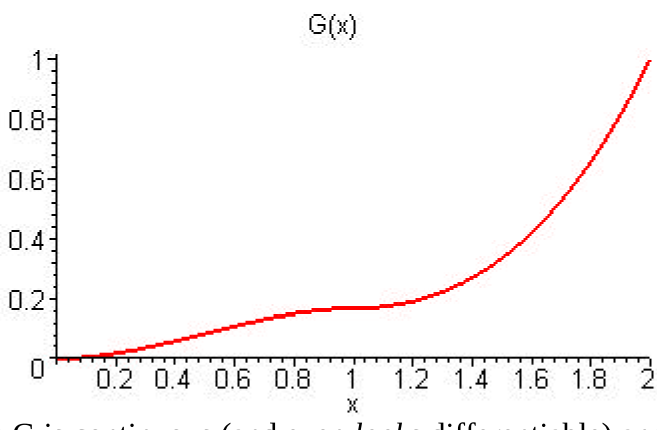
What has gone wrong is the fact that we (okay, *I*) have forgotten the definition of an *antiderivative of f on [a,b]*. Recall that F is an antiderivative of f if and only if $F'(x) = f(x)$ for all $x \in (a,b)$. Clearly, if my F is a true antiderivative of f then, since it is presumably differentiable on (0,2) it is continuous. A quick look at its graph should convince you that it is not!



Clearly, my F cannot be differentiable at x = 1. You should check that the **right hand derivative** of F at x = 1 is $F_+'(x) = \lim_{x \to 1^+} \frac{F(x) - F(1)}{x-1} = 0$ and the **left hand derivative** of F is $F_-'(x) = \lim_{x \to 1^-} \frac{F(x) - F(1)}{x-1} = -\infty$. This shows conclusively that *my* choice of F is not correct.

All we need do is transform F into a continuous function by adjusting the pieces so that they meet at x = 1. An easy and quick way would be to define the new function

$$G(x) = \begin{cases} \frac{x^2}{2} - \frac{x^3}{3} & x \in [0,1] \\ -\frac{x^2}{2} + \frac{x^3}{3} + \frac{1}{3} & x \in (1,2] \end{cases}$$ which is picture below.



This G is continuous (and even *looks* differentiable) and satisfies $G'(x) = f(x)\ \forall\ x \in (0,2)$. In other words it is a true antiderivative of f and can be used to get

Area $= \int_0^2 f(x)dx = G(2) - G(0) = 1 unit^2$ which we know is the **correct** answer.

<u>Moral of the story</u>: choose your friends wisely – be careful who you call an antiderivative.