The Average Value of f(x) on [a,b] and the Mean Value Theorem for Integrals

1. Average Value

Suppose that *f* is a continuous function on [a,b] and that you would like to calculate the average value of *f* on that interval. One way to start would be to sample some numbers from the interval and calculate their function values and take the average. If you choose many numbers from the interval this should give you a fairly good estimate of the average value of *f*. Let's do it by dividing [a,b] into n equal subintervals and using the right endpoints of the intervals for sample points. We would obtain:

$$AverageValue \approx \frac{f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_n)}{n} = \frac{\sum_{i=1}^{n} f(x_i)}{n} = \sum_{i=1}^{n} \frac{f(x_i)}{n}$$

Now, if we let $\Delta x = \dfrac{b-a}{n}$ then $n = \dfrac{b-a}{\Delta x}$ and we get that

$$AverageValue \approx \sum_{i=1}^{n} \frac{f(x_i)}{\frac{b-a}{\Delta x}} = \sum_{i=1}^{n} \frac{f(x_i)\Delta x}{b-a} = \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

. This makes it natural to define **the** average value of *f* on [a,b] by the following:

$$AverageValue = \lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

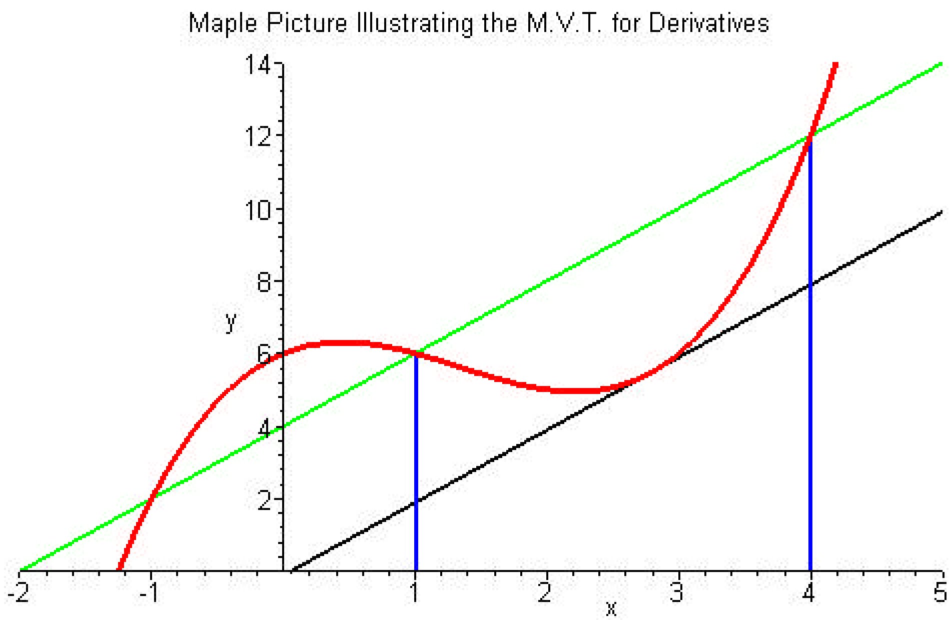
$$= \frac{1}{b-a}\int_a^b f(x)dx$$

<u>example</u>: The average value of $f(x) = \frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 6$ on the interval [1,4] can be calculated: $\dfrac{1}{4-1}\displaystyle\int_1^4 \left(\frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 6\right)dx = \frac{51}{8} = 6.375$

2. Mean Value Theorem

In Cal I you learned (of course you did!) the mean value theorem: *"If f is continuous on [a,b] and differentiable on (a,b) then there exists a number c,* $c \in (a,b)$*, such that* $f'(c) = \dfrac{f(b)-f(a)}{b-a}$. Geometrically, this theorem states that if a function is differentiable on (a,b) then there is at least one point on $y = f(x)$ between $(a, f(a))$ and

$(b, f(b))$ where the tangent line to $y = f(x)$ is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$. See the illustration below illustrating this for the function $f(x) = \frac{1}{2}x^3 - 2x^2 + \frac{3}{2}x + 6$ on the interval [1,4].



Now for some **Cal II**. Let $F(x) = \int_a^x f(t)dt$ where *f* is continuous on [a,b] and $x \in (a,b)$. By the *Fundamental Theorem of Calculus Part 1*, *F* is differentiable on (a,b) with it's derivative given by $F'(x) = f(x)$ and so satisfies the hypothesis of the Mean Value Theorem for derivatives. Therefore, there exists a number $c \in (a,b)$ such that:

$$F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{\int_a^b f(t)dt - \int_a^a f(t)dt}{b-a} = \frac{1}{b-a}\int_a^b f(t)dt \text{ and so} \quad .$$

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx \text{ since } F' = f \text{ and t is a "dummy" variable}$$

<u>One interpretation</u>: Since $\frac{1}{b-a}\int_a^b f(x)dx$ is also the average value of *f* on [a,b], this says that a continuous function on an interval will always assume its average value at least once somewhere in the interval.

<u>Another Interpretation</u>: If you rewrite the equation above as $\int_a^b f(x)dx = f(c)\cdot(b-a)$ then we have the "squaring of the integral." That is, it will always (for differentiable functions) be possible to find a rectangle whose base is the

interval [a,b] which has exactly the same area as that under $y = f(x)$ above [a,b]. See below for an illustration of this using the same function as above. The Maple File from which the graphs are taken will be found in my Cal2_Science folder on the Teacher Drive on the school network.