The Circumference & Area of a Circle: a Problem

Prove, without the use of any trigonometric functions, that there is a constant, say π, such that the area A and the circumference C of a circle of radius r are given by $A = p\mathrm{r}^2$ and $C = 2pr$.

Solution:

Consider a circle of radius r with equation $x^2 + y^2 = r^2$. Using Riemann sum arguments, not involving trigonometry, we know that:

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx \text{ and } C = 4\int_0^r \sqrt{1 + \left(\frac{d}{dx}\sqrt{r^2 - x^2}\right)^2}\,dx = 4r\int_0^r \frac{1}{\sqrt{r^2 - x^2}}\,dx \text{ where}$$

A and C are the area and circumference respectively.

Using the change of variable $z = \dfrac{x}{r}$ these formulae become:

$$A = 4r^2\int_0^1 \sqrt{1 - z^2}\,dz \quad \text{and} \quad C = 4r\int_0^1 \frac{1}{\sqrt{1 - z^2}}\,dz \ .$$

We now apply the *cheap trick* that $1 = (1 - z^2) + z^2$ combined with *integration by parts* (and several steps in the right direction!) to obtain: $\int_0^1 \frac{1}{\sqrt{1 - z^2}}\,dz = 2\int_0^1 \sqrt{1 - z^2}\,dz$.

Now, since $f(z) = \sqrt{1 - z^2}$ is a continuous function on [0,1] the F.T.C. guarantees the existence of the definite integral as a finite number (constant) so we can call $4\int_0^1 \sqrt{1 - z^2}\,dz$ by the name p to represent this constant and thus obtain:

$$C = 4r\int_0^1 \frac{1}{\sqrt{1 - z^2}}\,dz = 8r\int_0^1 \sqrt{1 - z^2}\,dz = 2\left(4\int_0^1 \sqrt{1 - z^2}\,dz\right)r = 2pr \text{ and}$$

$$A = \left(4\int_0^1 \sqrt{1 - z^2}\,dz\right)r^2 = pr^2 \ !$$

Very cool. And no trigonometry.