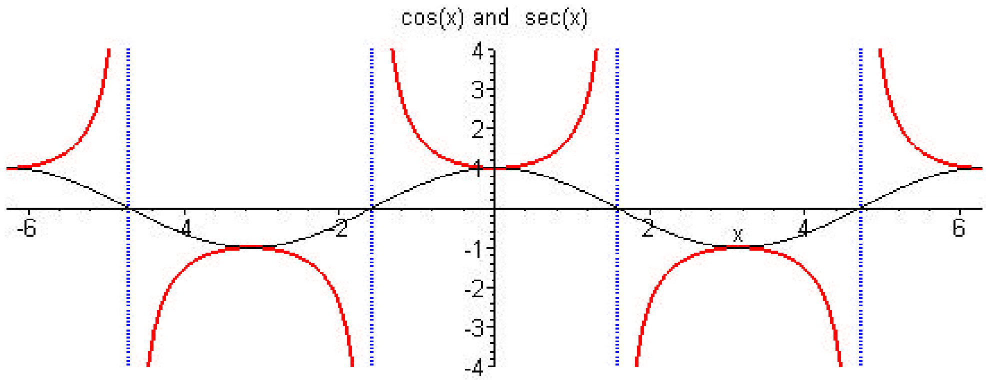
The Inverse Secant Function – Three Times

Have a look at the plots of the cosine and secant functions given below. It should be clear that in order to define an inverse for secant we must restrict its domain (just as for the sine, cosine and tangent functions). What is less clear than for the other functions is just *how* to restrict the domain.

1. Arcsecant Using $[0,p]-\{p/2\}$ *for the Domain of Secant.*

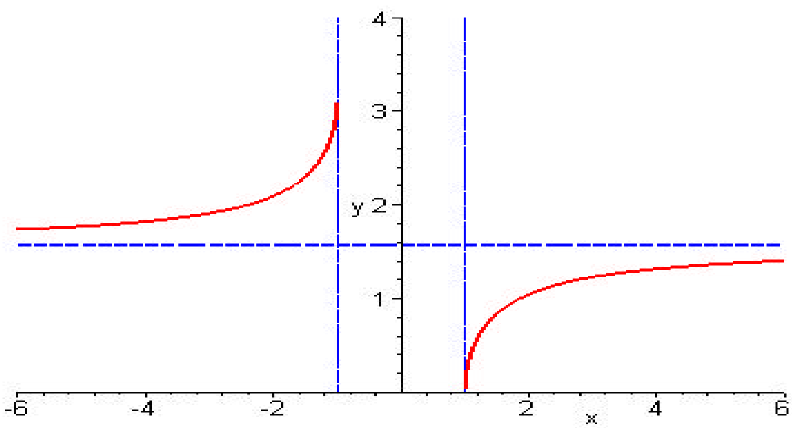
It seems to me, and to many other people, that the most natural (or obvious) choice is to restrict x to the interval $[0,p]-\{p/2\}$. Having done this,



$\sec(x)$ is now one-to-one and thus has an inverse. The inverse is defined in the usual way:

$$y=\sec^{-1}x \Leftrightarrow \sec y = x \quad \forall x\in[0,p]-\{p/2\}, \forall y\in(-\infty,-1]\cup[1,\infty)$$

See the graph below. It shows $y=\sec^{-1}x$ together with the lines $y=p/2$,

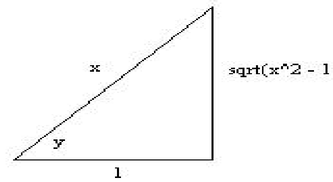


$x=-1$ and $x=1$.

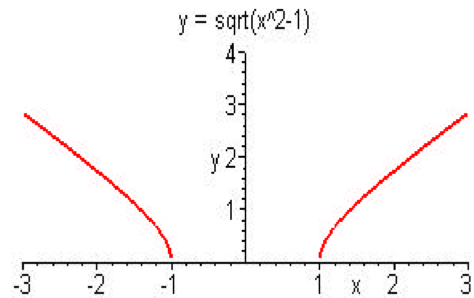
To find the derivative of the inverse secant we proceed via implicit differentiation: Let $y=\sec^{-1}x$ so that $\sec y = x$ and it follows that

$$\frac{d}{dx}\sec y = \frac{d}{dx}x$$
$$\sec y \tan y \cdot y' = 1$$
$$y' = \frac{1}{\sec y \tan y} = \frac{1}{|x|\sqrt{x^2-1}}$$

To see where the absolute value sign comes from, consider the triangle below. The sides are determined by the relationship $\sec y = x = \frac{x}{1}$. But you must keep in mind what is *y* in fact. It is the number (or angle in radians) $\sec^{-1} x$ and as such represents a value in the interval $[0,p]-\{p/2\}$ which depends on the choice of *x*. If $x \in (-\infty,-1]$ then $y \in (p/2, p)$ (i.e. a quadrant II angle) and so $\sec y \tan y$ will be positive since both tangent and secant are negative in QII. If $x \in [1,\infty)$ then $y \in [0, p/2)$ (i.e. a quadrant I angle) and so $\sec y \tan y$ is again positive. It follows that $y' > 0$ for all *x* in the domain of $\sec^{-1} x$ and to ensure this we take the absolute value of *x* in the derivative.



Sample Calculation: One important effect of the domain choice for the arcsecant function is in the calculation of some definite integrals. Consider the calculation $\int_{-2}^{-1} \sqrt{x^2-1}\,dx$ which requires trig substitution for its calculation. We will use the substitution $x = \sec q$ for $q \in [0,p]-\{p/2\}$. To change the limits of integration requires the inverse secant function:



$(x=-2) \Rightarrow (\sec q = -2) \Rightarrow \left(q = \sec^{-1}(-2) = \frac{2p}{3} \right)$ and $(x=-1) \Rightarrow (\sec q = -1) \Rightarrow q = \sec^{-1}(-1) = p$. The choices for q are because q must be a quadrant two angle because of the domain $[0,p]-\{p/2\}$.

The following calculations then yield:

$$\int_{-2}^{-1} \sqrt{x^2-1}\,dx = \int_{2p/3}^{p} \sqrt{\tan^2 q}\, \sec q \tan q\, dq = \int_{2p/3}^{p} -\tan q \cdot \sec q \tan q\, dq$$

$$= \int_{p}^{2p/3} \left(\sec^3 q - \sec q\right) dq = \frac{1}{2}\left[\sec q \tan q - \ln\left|\sec q + \tan q\right|\right]_{p}^{2p/3} .$$

$$= \sqrt{3} - \frac{1}{2}\ln\left(2+\sqrt{3}\right) \approx 1.07$$

You should make sure that you can fill in the significant number of missing steps in the calculation. Also, Maple gives the answer as $\sqrt{3} + \frac{1}{2}\ln\left(2-\sqrt{3}\right)$ and you should verify that the two answers are the same.

Important: the choice $\sqrt{\tan^2 q} = -\tan q$ is made because of the fact that the variable q is in quadrant II and hence $\tan q < 0$. If you rework the example using $\sqrt{\tan^2 q} = \tan q$ then you will get the answer $-\sqrt{3} + \frac{1}{2}\ln\left(2+\sqrt{3}\right) \approx -1.07$. This cannot be right as we recognize that the integral must be positive since the original function is positive.

2. *Arcsecant using* $[0, p/2) \cup [p, 3p/2)$ *for the Domain of Secant.*

With this new choice for the domain of $y = \sec x$ (again making it one-to-one) the formula for the derivative changes. The same approach as before (using the same triangle) yields that if $y = \sec^{-1} x$ then $y' = \dfrac{1}{\sec y \tan y} = \dfrac{1}{x\sqrt{x^2 - 1}}$. To see why the absolute value sign has disappeared consider the angle (number) $y = \sec^{-1} x$. If $x \in (-\infty, -1]$ then y is will be a QIII angle (number) and so $\sec y \tan y$ will be negative, as will $\dfrac{1}{x\sqrt{x^2-1}}$. If $x \in [1, \infty)$ then y will be a QI angle (number) and so, again, $\sec y \tan y$ and $\dfrac{1}{x\sqrt{x^2-1}}$ will agree.

Clearly, the graph of this version of $y = \sec^{-1} x$ must be different. It is decreasing on $x \in (-\infty, -1]$ and $\lim\limits_{x \to -\infty} \sec^{-1} x = 3p/2$ while $\lim\limits_{x \to -1} \sec^{-1} x = p$. On $x \in [1, \infty)$, the graph is increasing starting at (1,0) and satisfying $\lim\limits_{x \to \infty} \sec^{-1} x = p/2$.

<u>Sample Calculation Revisited</u> We can now try $\int_{-2}^{-1} \sqrt{x^2 - 1}\,dx$ again using this second definition of $y = \sec^{-1} x$. Under the same substitution approach the limits of integration will change as follows:

$$\int_{-2}^{-1} \sqrt{x^2 - 1}\,dx = \int_{4p/3}^{p} \sqrt{\tan^2 q}\, \sec q \tan q \, dq = \int_{4p/3}^{p} \tan q \sec q \tan q \, dq$$

. Note that $\sqrt{\tan^2 q} = \tan q$ since now q is a QIII angle and tangent is positive there. For practice, you should work through the details and make sure that you end up with **the same final result** as earlier.

3. *Yet Another way to Define Arcsecant*

Let $f(x) = \sec x$ for $x \in [0, p] - \{p/2\}$ and $g(x) = \cos^{-1}\left(\dfrac{1}{x}\right),\ x \neq 0$. **Recall:** f and g are inverses of each other if and only if $f(g(x)) = x\ \forall\ x \in D_g$ and $g(f(x)) = x\ \forall\ x \in D_f$.

$$f(g(x)) = \sec\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{1}{\cos\left(\cos^{-1}\left(\frac{1}{x}\right)\right)}$$

$$= \frac{1}{\frac{1}{x}} = x \text{ if } \frac{1}{x} \in [-1, 1] \text{ (which it is)}$$

$$g(f(x)) = \cos^{-1}\left(\frac{1}{\sec x}\right) = \cos^{-1}(\cos x).$$

$$= x \text{ if } x \in [0, p] \text{ (which it is)}$$

It follows that $\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$.

This makes deriving the derivative quite easy. Regard:

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{d}{dx}\left(\cos^{-1}\left(\frac{1}{x}\right)\right) = \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{\frac{x^2\sqrt{x^2-1}}{\sqrt{x^2}}} = \frac{1}{\frac{x^2}{|x|}\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}.$$