Logistic Growth Problem

The world population was ≈ 1.608 billion in 1900 and ≈ 2.517 billion in 1950. Predict the world population in 1992 based on these numbers and the assumption of **logistic growth** with a world population ceiling of 100 billion.

Let $P = P(t)$ be the population (in billions) at time *t* years with *t* = 0 corresponding to 1900 and let $P_m = 100$ be the maximum possible population. The **differential equation** that models the situation is given by:

$\frac{dP}{dt} = kP\left(P_m - P\right)$ where $k > 0$ is the proportionality constant. This equation is the result of the assumption that **the rate of growth** of *P* will be **jointly proportional** to both the population at time *t* and the **difference** between the population at time *t* and the maximum possible population. (Since we assume the population *P* is increasing it must be that $P' > 0$ so we must have that *k* > 0 since both P and P_m are positive.)

Separating the variables and writing the equation in differential form the solution can be written in the implicit form $\int \frac{1}{P\left(P_m - p\right)} dP = \int ktdt$. Using the partial fractions technique we get $\frac{1}{P_m}\int \left(\frac{1}{P} + \frac{1}{P_m - P}\right) dP = \int ktdt$. Integrating both sides yields:

$$\frac{1}{P_m}\left[\ln P - \ln\left(P_m - P\right)\right] = kt + C_0$$

$$\ln\left(\frac{P}{P_m - P}\right) = P_m kt + C_1 \text{ where } C_1 = P_m C_0$$

$$\frac{P}{P_m - P} = e^{P_m kt + C_1} = Ke^{rt} \text{ where } K = e^{C_1},\ r = P_m k$$

Solving this last equation for P (make sure **you** can do it also) yields:

$$P = \frac{P_m Ke^{rt}}{1 + Ke^{rt}} = \frac{P_m K}{K + e^{-rt}} = \frac{P_m}{1 + K'e^{-rt}} \text{ where } K' = \frac{1}{K}. \qquad (*)$$

To check the long term behaviour of the model we can note that $\lim\limits_{t\to\infty} P = \frac{P_m}{1 + K'\lim\limits_{t\to\infty} e^{-rt}} = \frac{P_m}{1+0} = P_m$ where we recognize that since $r = P_m k > 0,\ e^{-rt} \to 0$ as $t \to \infty$. This jives with our intuition that in the long run, the population will get closer and closer to the maximum sustainable population.

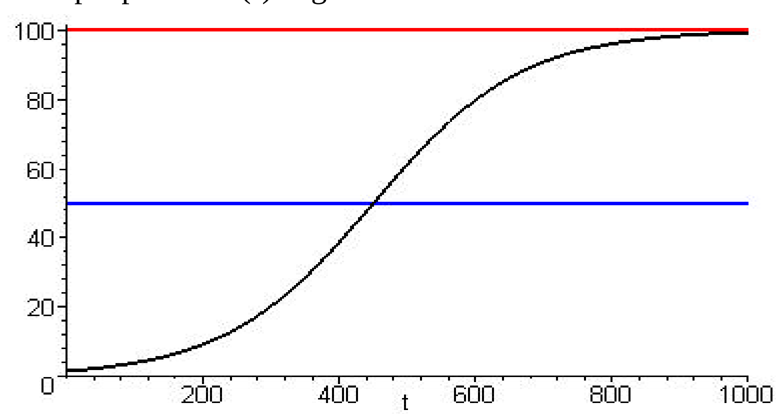
Before going to the concrete numbers of the particular problem let's play around a bit. Let $P(0)=P_0$ be the initial (starting) population. "Plugging in" we obtain $P_0=\frac{P_m}{1+K'}$ so that $K'=\frac{P_m}{P_0}-1$. Putting this value back into (*) we get $P=\frac{P_m}{1+\left(\frac{P_m}{P_0}-1\right)e^{-rt}}$ and this can be rearranged (multiply top and bottom by P_0 and then divide top and bottom by P_m) to obtain $P=\frac{P_0}{\frac{P_0}{P_m}+\left(1-\frac{P_0}{P_m}\right)e^{-rt}}$. In this form it can be seen that if P_m is **VERY BIG** in relation to P_0 then $\frac{P_0}{P_m}\approx 0$ and in this situation we obtain $P\approx\frac{P_0}{e^{-rt}}=P_0e^{rt}$ which you can check (by solving the differential equation $\frac{dP}{dt}=rP$) is the **exponential growth** model with initial population P_0 and growth rate *r*. Crazy.

Even crazier. Consider the graph of $P=P(t)$. By studying its derivatives we discover that it starts concave up and then becomes concave down. Hence, it has an **inflection point**. At this point, the **rate** at which P has been increasing starts to **slow down** (think about it). In other words, the time of quickest growth occurs at the inflection point. The *t* coordinate of the point must be where $P''(t)=0$ and you should be able to confirm that at this particular time $P=\frac{P_m}{2}$! And doesn't that just **make sense** when you think about it for a minute?!

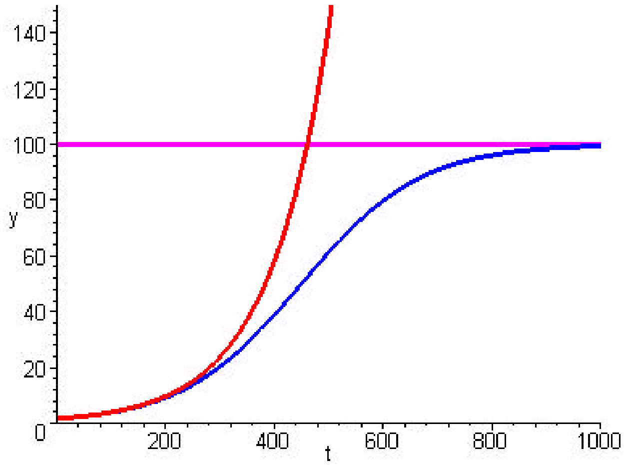
Okay, time to tackle the concrete problem. If we use the solution in the form of (*) then we know that $K'=\frac{100}{1.608}-1\approx 61.18905$.

Now use the fact that $P(50)=2.517$ to obtain $r\approx 0.00914716$. The model then becomes $P(t)\approx\frac{100}{1+61.18905e^{-0.00914716t}}$. It follows that, according to this model the population in 1992 should have been $P(92)\approx 3.653$ billion people.

It is of interest to check out the Maple plot of $P(t)$ together with the horizontal lines $P=P_m=100$ and $P=\frac{P_m}{2}=50$. According to the numbers, the rate of increase will not begin to slow until $t\approx 450$ which translates into the year 2350.



Is there a significant difference if we use an exponential model? To find out we can repeat the calculations with the differential equation $\frac{dP}{dt} = kP$ which has the general solution $P = P(t) = Ke^{kt};\ K > 0, k > 0$. Using the same population figures the model becomes $P(t) \approx 1.608e^{0.0089615t}$. The prediction for 1992 is then $P(92) \approx 3.667$ billion which is only slightly more than the first estimate. Recall how the theory says that if the difference is great between the population and the ceiling population then the two models are approximately equal. To underline this take a look at the accompanying plot of the two models.



According to the plot, there will be no significant difference between the two until about 200 years after 1900 (i.e. post 2200 AD).