Separable Differential Equations

Any equation that can be put in the form $\frac{dy}{dx} = f(y)g(x)$ is a separable differential equation. Such an equation can be solved by "separating the variables" and integrating both sides of the result as in the following example.

Problem 1. Find the solution of the equation $\frac{dy}{dt} = te^y$ (*) that satisfies the condition $y(1) = 0$ (which means that when $t = 1$, $y = 0$).

Solution: $\frac{dy}{dt} = te^y \Leftrightarrow e^{-y}dy = tdt$

$$\Leftrightarrow \int e^{-y}dy = \int tdt$$

$\Leftrightarrow -e^{-y} = \frac{t^2}{2} + C$. This would be the general solution to (*) and could (in this case) be solved explicitly for y (yielding $y = y(t)$) or left in its implicit form. Here though we apply the condition $y(1) = 0$ to obtain the particular solution of (*) that satisfies the given initial condition:

$y(1) = 0 \Leftrightarrow -e^0 = \frac{1^2}{2} + C$ so $C = -1 - \frac{1}{2} = -\frac{3}{2}$ and the sought after solution is therefore $-e^{-y} = \frac{t^2}{2} - \frac{3}{2}$ or $\boxed{t^2 = 3 - 2e^{-y}}$. If an explicit answer is desired (which is not always easy, or possible, this can be manipulated into $y = \ln 2 - \ln\left(3 - t^2\right)$.

Exponential Growth/Decay Suppose that the rate of change of a quantity y with respect to time (its rate of growth or decay) is directly proportional to the amount present at time t. This translates into the (separable) differential equation $\frac{dy}{dt} = ky$ where k is a constant (the proportionality constant) and $y = y(t)$ is the amount present at time t.

We can solve this general problem as follows:

$$\frac{dy}{dt} = ky, \quad y > 0$$

$$\Leftrightarrow \frac{1}{y}dy = kdt$$

$$\Leftrightarrow \int \frac{1}{y}dy = \int kdt \Leftrightarrow \ln y = kt + C$$

$$\Leftrightarrow y = e^{kt+C} = e^C e^{kt}$$

$$y\left(t\right) = Ke^{kt} \quad \text{(where } K = e^{kt}\text{)}$$

The values of K and k will de determined by the conditions of the particular problem.

This final general solution $\frac{dy}{dt} = ky$ is the exponential growth (or decay) model and can be used to describe certain physical situations such as

(1) the growth of a bacterial culture in an "ideal" environment (unlimited space and infinite resources) or

(2) the decay of a radioactive element.

Test yourself by doing problems **#3 and #9** on page 620.

Problem 2. Suppose an investment grows at a rate that is proportional to the amount present at any given time. $1000 is invested initially and doubles in 10 years. Find the amount present at time *t* years and determine how many years it will take for the investment to triple in size. To **test yourself** you should try to solve this yourself first and *then* look at my solution.

Solution Let $A = A(t)$ be the amount ($) present at time *t* years. The situation is then described by the differential equation $\frac{dA}{dt} = rA$ together with the initial conditions $A(0) = 1000,\ A(10) = 2000$. The general solution is found exactly as above and yields $A(t) = Pe^{rt}$ where *P* and *r* are constants. We now apply the conditions to find *P* and *r*: $A(0) = 1000 \Rightarrow 1000 = Pe^0 = P$ so $P = 1000$. (The letter *P* was chosen because $1000 is the **P**rincipal investment.) The model is now $A(t) = 1000e^{rt}$ and the second condition can be applied: $A(10) = 2000 \Leftrightarrow 2000 = 1000e^{10r} \Leftrightarrow e^{10r} = 2 \Leftrightarrow r = \frac{\ln 2}{10}$ ($r = \ln 2 / 10 \approx 0.07$ and this corresponds to the interest rate of r = 7% per annum). The full model is therefore $\boxed{A(t) = 1000e^{\frac{\ln 2}{10}t}}$ which (see if you can do it!) can also be written as $A(t) = 1000 \cdot 2^{\frac{t}{10}}$. To determine the tripling time we simply solve the equation $A(t) = 3000$ for *t*: $1000e^{\frac{\ln 2}{10}t} = 3000 \Leftrightarrow e^{\frac{\ln 2}{10}t} = 3 \Leftrightarrow \frac{\ln 2}{10}t = \ln 3$

$$\Leftrightarrow t = \frac{10\ln 3}{\ln 2} \approx 15.8\, years.$$

Therefore it will take almost 16 years to triple the principal.