ELEMENTARY MATRICES

Definition: E is an **elementary matrix** if E was (or could be) produced by performing a **single E**lementary **R**ow **O**peration (ERO) on an identity matrix I.

It is of utmost importance in this context to recall that there are **only** three ERO's possible:

1) $kR_i + R_j \to R_j$ (add a multiple of one row to another row)
2) $kR_i \to R_i$ (multiply any row by a **nonzero** scalar)
3) $R_i \leftrightarrow R_j$ (interchange two rows)

example 1 : $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix since it can be produced from I by a single ERO as follows: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

example 2: $E_2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix since it can be produced from I by a single ERO as follows: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{3R_1 \to R_1} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Example 3: $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ is an elementary matrix since it can be produced from I by a single ERO as follows: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Let B be any matrix, say $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Now, perform a single ERO of type (1) on B: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow{-2R_1 + R_2 \to R_2} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{pmatrix}$. Now, let E_1 be the elementary matrix that is produced by doing the same ERO on the 3x3 identity matrix: $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(as in example 1) and consider the matrix product E_1B: E_1B =

$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{pmatrix}$. It should be clear that multiplying B by E_1 **on the left** has the same effect as the original row operation on B.

Now lets do an ERO of type 2 on the matrix E_1B above:

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{3R_1 \to R_1} \begin{bmatrix} 3 & 6 & 9 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$.

We now try multiplying the matrix E_1B **on the left** by the elementary matrix produced by the same ERO (which is exactly the elementary matrix of example 2):

$E_2\left(E_1B\right) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix}$. Notice that multiplication by E_2 has had the same effect as the ERO that produced E_2.

One last kind of ERO to try, namely, one of type 3. We will do it on the last matrix above, that is, on $E_2\left(E_1B\right)$.

$\begin{bmatrix} 3 & 6 & 9 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 3 & 6 & 9 \\ 7 & 8 & 9 \\ 2 & 1 & 0 \end{bmatrix}$.

The elementary matrix E_3 of example 3 was produced by this same ERO and we now observe the effect of multiplying $E_2\left(E_1B\right)$ **on the left** by it:

$E_3\left(E_2\left(E_1B\right)\right) = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 6 & 9 \\ 2 & 1 & 0 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 7 & 8 & 9 \\ 2 & 1 & 0 \end{bmatrix}$. Again, you can see that left multiplication by E_3 has the same effect on $E_2\left(E_1B\right)$ as doing the ERO.

The moral of the story: every ERO can be accomplished by left multiplication by the elementary matrix that was produced by the same ERO. ERO's are really the same as left multiplication by elementary matrices.

Big Example: Finding A^{-1} using elementary matrices.

Consider the matrix $A = \begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. We can put A into RREF using ERO's as follows:

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \leftrightarrow R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[\frac{1}{3}R_2 \to R_2]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \xrightarrow[2R_2 + R_3 \to R_3]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

. The three ERO's used can be encoded into elementary matrices as follows: the first ERO can be done by $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, the second by $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and the third one by $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Each of these elementary matrices were produced by performing the same ERO that was used in each step of the reduction on an identity matrix.

It follows that the row reduction of A to I can be accomplished by successive multiplications on the left by these three elementary matrices:

$E_3\left(E_2\left(E_1 A\right)\right) = I$ and then the brackets can be rearranged to produce $\left(E_3 E_2 E_1\right) A = I$. Since $B = E_3 E_2 E_1$ is just another matrix, we have that $BA = I$ and it must be that $B = A^{-1}$.

To see if this can really be true all we need to do is the multiplication. You first should check that $B = E_3 E_2 E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & 0 \\ 1 & 2/3 & 0 \end{bmatrix}$ by direct calculation. Next you should verify that $BA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/3 & 0 \\ 1 & 2/3 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Finally, check yourself that $AB = I$ also. The conclusion is inescapable: $B = A^{-1}$.

Practice Problem: Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{bmatrix}$ and find A^{-1} by the same procedure. Reduce A to I using **only** ERO's (it can be done in three steps) and then create the elementary matrices that perform those same ERO's and proceed as in the above Big Example.