Inverses of Elementary Matrices

Recall that an **elementary matrix E** is any matrix produced from an identity matrix by a **single** E.R.O. Further, multiplying another matrix **A** by **E** on the left has the effect of performing the E.R.O. on **A** that was used on **I** to produce E.

Now, it should be clear that every E.R.O. can be reversed, or **undone**, by another E.R.O. The elementary matrix that is formed by doing the "undoing" E.R.O. on **I** will be the inverse of the original elementary matrix. See the table below.

E.R.O. that produces E	**Undoing E.R.O. (and produces** E^{-1}**)**
$kR_i + R_j \to R_j$	$-kR_i + R_j \to R_j$
$kR_i \to R_i$	$\frac{1}{k}R_i \to R_i$
$R_i \leftrightarrow R_j$	$R_i \leftrightarrow R_j$

Suppose A is a matrix that can be reduced to I in three E.R.O.'s. Let E_1, E_2 and E_3 be the three elementary matrices that performed the first, second and third of the E.R.O.'s. (If you can't remember how to do that then refer to the big example in the *Elementary Matrices and Finding A Inverse* notes on this website.)

It follows that $E_3E_2E_1A = I$. If we multiply both sides of this equation by E_3^{-1} (from the left) we get $E_3^{-1}E_3E_2E_1A = E_3^{-1}I$ so that $E_2E_1A = E_3^{-1}$ (since $E_3^{-1}E_3 = I$). Now multiply both sides by E_2^{-1} and obtain $E_1A = E_2^{-1}E_3^{-1}$. Finally, multiply both sides by E_1^{-1} and obtain at last that $A = E_1^{-1}E_2^{-1}E_3^{-1}$. Since E_1^{-1}, E_2^{-1} and E_3^{-1} are all elementary matrices, we have written A as a product of elementary matrices. This example, using three elementary matrices, can be easily generalized to the following theorem.

Theorem: *If **A** is invertible then A can be written as a product of elementary matrices.*

Big Example. Write $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \end{pmatrix}$ as a product of elementary matrices.

Solution: We first reduce A to I by E.R.O.'s , keeping track of the E.R.O.'s.

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & -1/3 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{-1/3R_3 \to R_3} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{-2R_2 + R_1 \to R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The E.R.O. that undoes the first E.R.O. is $-3R_3 \to R_3$ and the elementary matrix that corresponds to this is $E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$. To undo the second E.R.O. we do $2R_2 + R_1 \to R_1$ and the elementary matrix that corresponds to this is $E_2^{-1} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. The last E.R.O., $R_2 \leftrightarrow R_3$, is undone by simply repeating it and the E.R.O. that corresponds to this is $E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Now, using the fact that if $E_3 E_2 E_1 A = I$ then $A = E_1^{-1} E_2^{-1} E_3^{-1}$ we get that $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and we have written *A* as the product of three elementary matrices. You should check for yourself that the product actually does result in the matrix *A*!