Two Economic Models of Vassily Leontiff

(Notes abridged from Howard Anton's *Elementary Linear Algebra with Applications*.)

I **The Closed Model**

Setup: A *finite* number of industries (indexed 1, 2, 3.…) operating in a *closed* economy (i.e. nothing goes out or comes in from the "outside") is such a way that over a fixed period of time each industry produces an output that is *completely* consumed by the *n* industries.

Problem: Determine a price structure **P** so that each industry is in "equilibrium." That is, determine for *each* industry the price that industry should charge for its *total output* so that total expenditures equal total income for each industry.

Notation: Let p_i be the price charged by the i[th] industry for its *total output* over the fixed period in question. $\mathbf{P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ is the price vector or structure for the *n* industries.

Let e_{ij} be the *fraction* of the total output of industry *j* consumed by industry *i*. The matrix $\mathbf{E} = \lfloor e_{ij} \rfloor_{nxn}$ is called the *Exchange* or *Input-Output* matrix for the economy.

Notice: $e_{ij}p_j$ = the money spent by industry *i* to purchase the output it needs from industry *j*. It follows that :

$$e_{i1}p_1 + e_{i2}p_2 + e_{i3}p_3 + \cdots + e_{in}p_n = \text{the } \textit{total expenditures} \text{ of industry } i.$$

$$\text{and so } \mathbf{EP} = \begin{bmatrix} e_{11}p_1 + \cdots + e_{1n}p_n \\ e_{21}p_1 + \cdots + e_{2n}p_n \\ \vdots \\ e_{n1}p_1 + \cdots + e_{nn}p_n \end{bmatrix} = \begin{bmatrix} \text{total expenditure of industry 1} \\ \text{total expenditure of industry 2} \\ \vdots \\ \text{total expenditure of industry n} \end{bmatrix}.$$

Now, since p_i = the price charged by industry i for its total output, industry i will be in equilibrium if $e_{i1}p_1 + e_{i2}p_2 + e_{i3}p_3 + \cdots + e_{in}p_n = p_i$ (total expenditure of industry i = total income of industry i). It follows that for the *whole system* to be in equilibrium it must satisfy the *matrix equation* $\boxed{\mathbf{EP} = \mathbf{P}}$.

A little bit of matrix algebra yields: **EP =P** <=>**EP – P = 0** <=> **(E – I)P = 0** [(*)]

We all recall that a *homogeneous* linear system such as [(*)] above always has a solution: the trivial solution. We are interest in *nonzero* solutions and recall that (*) will have nontrivial solutions if and only if det(**E – I)** = 0.

By definition of **E**, and the condition that outputs are *completely* consumed, it follows that the columns of **E** *all* sum to one. Try this with a small (say a (3 x 3)) matrix **E** to see why. With this in mind it should be clear that all the columns of **E – I** sum to zero (just write out **E – I** and the column sums to see why) and it follows from this fact that det(**E – I**) = 0! We have proved the following:

Theorem: There are always nontrivial solutions to the equation **(E – I)P = 0**.

Further, it can be shown (not so easily as the above!) that since e_{ij} is nonnegative for all i and j, **EP = P** always has a nontrivial solution whose entries are all nonnegative.

Example: A carpenter, electrician and plumber (each owning their own company) decide to pool their services and renovate their three houses over their ten day holiday. For nefarious tax purposes they agree to pay each other a "reasonable" daily wage for work performed (even, of course, for work each does on his own home – after all, they are using their company resources) and do so in such a manner that each individual's expenditure equals their income over this period. That is, they form a *closed economy* over the holiday period. They reckon that the job can be accomplished by following the following schedule:

Days of work in home of	**Work Performed by**		
	Carpenter	Electrician	Plumber
Carpenter	2	1	6
Electrician	4	5	1
Plumber	4	4	3

Let the carpenter be industry 1, the electrician industry 2 and the plumber industry 3. From the info in the table the e_{ij}'s will form the following matrix:

$$\mathbf{E} = \begin{bmatrix} 2/10 & 1/10 & 6/10 \\ 4/10 & 5/10 & 1/10 \\ 4/10 & 4/10 & 3/10 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.4 & 0.3 \end{bmatrix}$$

The problem is to determine $\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ such that **EP = P** or, equivalently, **(E – I)P = 0**.

Writing out the matrices yields: $\begin{bmatrix} -0.8 & 0.1 & 0.6 \\ 0.4 & -0.5 & 0.1 \\ 0.4 & 0.4 & -0.7 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .and writing the systems associated *augmented matrix* and doing some number crunching will produce the following: $\begin{bmatrix} -.08 & 0.1 & 0.6 & 0 \\ 0.4 & -0.5 & 0.1 & 0 \\ 0.4 & 0.4 & -0.7 & 0 \end{bmatrix} >>> > > \begin{bmatrix} 1 & 0 & -31/36 & 0 \\ 0 & 1 & -8/9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and the general solution is $p_1 = \frac{31}{36}t$, $p_2 = \frac{8}{9}t$, $p_3 = t$. Now, if we replace t by 36S (to get rid of the nasty fractions) and write the solution in matrix (or vector) form the solution becomes: **P** **=** $S\begin{bmatrix} 31 \\ 32 \\ 36 \end{bmatrix}$. The parameter S is called the "scaling factor" for the economy.

If the three renovators decide that $100 per day (or $1000 for the 10 day period) is, for tax purposes, a "reasonable" wage then we can choose S = 30 to obtain **P =** $\begin{bmatrix} 930 \\ 960 \\ 1080 \end{bmatrix}$. In other words, a *daily* wage of $93, $96 and $108 for the carpenter, electrician and plumber respectively will result in equilibrium (each individual's total income will equal their total expenditures).

II **The Open Model**

Setup: In the open model, the industries distribute outputs amongst themselves (to keep operating) and also meet an *outside demand*.

Problem: In the open model, the prices of the outputs are *fixed* and the problem is to determine the *levels of output* that each industry must maintain to completely meet the outside demands. Contrast this to the *closed model* in which the outputs are fixed and the *price structure* is determined in order to out the system into equilibrium (so income = expenditures).

Notation: Let x_i = the money value of the *total output* of the i^{th} industry. The *vector* **x** of total outputs is called the *production vector* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Let d_i = the money value of the output of the i^{th} industry needed to satisfy the *outside demand*. The *demand vector* is $\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$.

Let c_{ij} = the money value of the output of the i^{th} industry needed by j^{th} industry to produce ***one unit*** of the money value of its own output. (For instance, if Hydro Quebec is industry 1 and the steel industry is industry 2 then c_{12} = \$0.10 means that the steel industry requires 10¢ of electricity for each \$1 of steel it produces and c_{21} = \$0.005 means that Hydro Quebec requires ½ ¢ worth of steel for each \$1 of electricity it produces. c_{11} = \$0.15 means that Hydro Quebec requires 15¢ worth of its own product for each \$1 it produces.) $\mathbf{C} = \lfloor c_{ij} \rfloor$ is called the *consumption matrix* (or input-output matrix) for the economy.

Consider an economy of 3 industries and look at the vector which is the result of the matrix multiplication **Cx**. The first entry is $c_{11}x_1 + c_{12}x_2 + c_{13}x_3$ and represents the value of the total output of the first industry required by *all* the industries to meet their production demands. Now, if the *outside demand* for the first industry's product is d_1 then $c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + d_1$ is the *total demand* on the first industry. Since we want the total production to equal the total demand, the requirement we place on the first industry must be: $c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + d_1 = x_1$. Similarly, we want the internal + external demands on each industry to equal that industry's output. In other words, we want the *matrix equation* **Cx + d = x** to be satisfied. This equation can be put into the equivalent form **(I – C)x = d** and this represents the *linear system* that must be solved for the unknown output levels (the x_i's) for the economy to exactly meet the outside demands on it.

<u>Example</u>: A town with three industries (mining = industry 1, power company = industry 2, railroad = industry 3) was surveyed and it was found that: for each \$1 of ore mined, the mining industry requires 25¢ of power and 25¢ worth of transportation. To produce \$1 of electricity, the power company needs 65¢ of coal from the mining company, 5¢ of its own power and 5¢ worth of transportaion. To provide \$1 of transportation the railrod company needs 55¢ of coal and10¢ of power. In a certain week the town receives orders for \$50,000 of coal and \$25,000 for electricity from "outside" (and no outside demands on the rail company). How much must each industry produce in that week to exactly meet their own and outside demands?

You should confirm that, for this problem, $\mathbf{C} = \begin{bmatrix} 0 & .65 & .55 \\ .25 & .05 & .10 \\ .25 & .05 & 0 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 50{,}000 \\ 25{,}000 \\ 0 \end{bmatrix}$. We require that $\mathbf{Cx} + \mathbf{d} = \mathbf{x}$ which becomes $(\mathbf{I} - \mathbf{C})\mathbf{x} = \mathbf{d}$. If you happen to have a high-powered machine on your desk you can compute $(\mathbf{I} - \mathbf{C})^{-1}$ and arrive at the conclusion: $\mathbf{x} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} = \frac{1}{503}\begin{bmatrix} 756 & 542 & 470 \\ 220 & 690 & 190 \\ 200 & 170 & 630 \end{bmatrix}\begin{bmatrix} 50{,}000 \\ 25{,}000 \\ 0 \end{bmatrix} = \begin{bmatrix} 102{,}087 \\ 56{,}163 \\ 28{,}330 \end{bmatrix}$. This translates into: "In order to meet the internal and external demands of the week, the mining company must produce \$102,087 worth of ore, the power company must output \$56,163 worth of spark and the railroad must provide \$28,330 worth of service."

Problems

1) Three neighbours have backyard vegetable gardens. Neighbours A, B and C grow tomatoes, corn and lettuce respectively. They agree to divide their crops among themselves as follows: A gets ½ of the tomatoes, $\frac{1}{3}$ of the corn and ¼ of the lettuce. B gets $\frac{1}{3}$ of the tomatoes, $\frac{1}{3}$ of the corn and ¼ of the lettuce. C gets $\frac{1}{6}$ of the tomatoes, $\frac{1}{3}$ of the corn and ½ of the lettuce. What prices should they assign to their respective crops if the lowest price is to be $100 and the conditions for a closed economy are met?

2) Three engineers – a civil engineer (CE), an electrical engineer (EE) and a mechanical engineer (ME) – each have a consulting firm. Their consulting is of a multidisciplinary nature and so they buy a portion of each other's services. For each $1 of consulting the CE does, she buys $0.10 of the EE's services and $0.30 of the ME's services. For each $1 of consulting the EE does, she buys $.20 of the CE's services and $0.40 of the ME's services. And for each $1 of consulting the ME does, she buys $0.30 of the CE's services and $0.40 of the EE's services. In a certain week the CE receives outside consulting orders of $500, the EE receives orders of $700 and the ME receives orders of $600.

3) An economy can be divide into 3 sectors: **M**(anufacturing), **S**(ervices) and **A**(griculture). In a certain year the economy was surveyed and the products were seen to be employed according to the following table ($ billions):

	consumer/buyer					
producer/ seller		**M**	**S**	**A**	**other demand**	**total**
	M	3	0.6	1	10.4	15
	S	1.5	1.8	2	0.7	6
	A	1.5	0.6	4	3.9	10

i) Set up the equations to determine what outputs of each sector are required if "other demand" changes to $33 billion for **M**, $8 billion for **S** and $16 billion for **A**.

ii) Solve the resulting system to find the new outputs necessary using Gauss Elimination.

4) Consider an economy with sectors **A** and **B**, represented in the following table (data are in $ millions):

	consumer/buyer				
producer/ seller		**A**	**B**	**other demand**	**total**
	A	120	40	140	300
	B	60	40	100	200

Determine the output of each sector if "other demand" changes to 210 for **A** and 147 for **B**. Solve the resulting system by using an *inverse matrix*.

5) A system composed of two industries, coal and steel, has the following inputs:

(a) To produce $1 worth of output, the coal industry reqires $0.10 of its own product and $0.80 of steel.

(b) To produce $1 worth of output the steel industry requires $0.10 of its own output and $0.20 of coal.

Find **C**, the input-output matrix (inter industry consumption matrix) for this system and solve for the outputs **X** if the outside demands are **D =** $\begin{bmatrix} 10{,}000 \\ 20{,}000 \end{bmatrix}$ **.**

6) An industrial system has two industries with the following inputs.

(a) to produce $1 of output, industry A requires $0.30 of its own product and $0.40 of industry B's product.

(b) to produce $1 of output, industry B requires $0.20 of is own product and $).40 of industry A's product.

Find **C,** the consumption (input-output) matrix for this system. Find the outputs **X** when the outside demands are given by **D** = $\begin{bmatrix} 50{,}000 \\ 30{,}000 \end{bmatrix}$.