Chebychev's Theorem: A Proof for the Continuous case.

Suppose that X is a *random variable* with probability density function (p.d.f.) f(x).

$$s^2 = E(x-m)^2 = \int_{-\infty}^{\infty} (x-m)^2 f(x)dx$$

$$= \int_{-\infty}^{x-ks} (x-m)^2 f(x)dx + \int_{x-ks}^{x+ks} (x-m)^2 f(x)dx + \int_{x+ks}^{\infty} (x-m)^2 f(x)dx$$

so $\sigma^2 \geq \int_{-\infty}^{x-k\sigma} (x-\mu)^2 f(x)dx + \int_{x+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$ since *all* the integrals are *non-negative* (why??).

$$\geq \int_{-\infty}^{\mu-k\sigma} k^2\sigma^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} k^2\sigma^2 f(x)dx$$ [1]

Dividing both sides by the positive $k^2\sigma^2$ it follows that

$$\frac{1}{k^2} \geq \int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx$$

Now, the first integral on the right is $P(x \leq \mu - k\sigma)$ and the second is $P(x \geq \mu + k\sigma)$ and so it follows that $P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$ and so, by considering the *complimentary* probability we conclude that:

$$P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}.$$

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[1] If $x \leq \mu - k\sigma$ then $(x-\mu)^2 \geq k^2\mu^2$ since $k\sigma \geq 0$. Similarly for $x \geq \mu + k\sigma$.