The Normal Distribution Curve is a P.D.F.

Gamma Function/Distribution

Let $f(y) = \begin{cases} ky^{a-1}e^{-y/b} & for \quad y > 0 \\ 0 & for \quad y \le 0 \end{cases}$ where á > 0, â > 0. For f to be a valid p.d.f. it must be that $\int_0^\infty f(y)dy = 1$.

$\Rightarrow k\int_0^\infty y^{\alpha-1}e^{-x/\beta}dy = 1$. Now, let $x = \frac{y}{\beta}$, $\beta dx = dy$ so that the equation becomes

$$k\int_0^\infty (\beta x)^{\alpha-1}e^{-x}\beta dx = 1$$

$$k\beta^\alpha \int_0^\infty x^{\alpha-1}e^{-x}dx = 1$$

$k\beta^\alpha\Gamma(\alpha) = 1$ where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ = the *Gamma Function* for á > 0.

Now, if we set $k = \frac{1}{\beta\Gamma(\alpha)}$ we get the p.d.f. of the *Gamma Distribution*:

$$f(y) = \begin{cases} \frac{1}{\beta^\alpha\Gamma(\alpha)} y^{\alpha-1}e^{-y/\beta} & for \;\; y > 0 \\ 0 & for \;\; y \le 0 \end{cases}.$$

Fact: Ã(á) = (á – 1)Ã(á – 1) for á > 1.

Proof: $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dy = \lim_{m\to\infty}\int_0^\infty x^{a-1}e^{-x}dx$

Let $I = \int x^{a-1}e^{-x}dx$ and use *Integration by Parts* with: $\begin{matrix} u = x^{a-1} & dv = e^{-x}dx \\ du = (a-1)x^{a-2}dx & v = -e^{-x} \end{matrix}$ to obtain $I = -x^{a-1}e^{-x} + (a-1)\int x^{a-2}e^{-x}dx$ and so

$$\Gamma(a) = \lim_{m\to\infty}\left(-x^{a-1}e^{-x}\Big|_0^m\right) + (a-1)\int_0^\infty x^{a-2}e^{-x}dx$$ ♦

$$= 0 + (\alpha-1)\Gamma(a-1) = (\alpha-1)\Gamma(a-1) \qquad \text{Q.E.D.}$$

Now, if á = 1 we get $\Gamma(1) = \int_0^\infty e^{-y}dy = 1$ (Check!!)

so that Ã(1) = 1. It follows with the result before that if n is a **N**atural number then:

Ã(n) = (n – 1)Ã(n – 1)
= (n – 1)(n – 2)Ã(n – 2)
= (n – 1)(n – 2)(n – 3)Ã(n –3)

and eventually you arrive at:

Ã(n) = (n – 1)(n – 2)(n – 3) ... (3)(2)Ã(1) = (n – 1)!

That is: Ã(n) = (n – 1)!, or, if you prefer, n! = Ã(n + 1).

$$\boxed{\textit{If } n \in N \textit{ then } \Gamma(n) = (n-1)!}$$

Consider the integral $2^{1-a}\int_0^\infty z^{2a-1}e^{-\frac{1}{2}z^2}\,dz$ for $a > 0$ and use the substitution $z^2 = 2x$, $zdz = dx$ to obtain:

$$2^{1-\alpha}\int_0^\infty z^{2\alpha-1}e^{-\frac{1}{2}z^2}\,dz = 2^{1-\alpha}\int_0^\infty z^{2\alpha-2}e^{-\frac{1}{2}z^2}\,zdz = 2^{1-\alpha}\int_0^\infty \left(z^2\right)^{\alpha-1}e^{-\frac{1}{2}z^2}\,zdz$$

$$= 2^{1-\alpha}\int_0^\infty (2x)^{\alpha-1}e^{-\frac{1}{2}2x}dx = 2^{1-\alpha}2^{\alpha-1}\int_0^\infty x^{\alpha-1}e^{-x}dx$$

$$= \int_0^\infty x^{\alpha-1}e^{-x}dx = \Gamma(\alpha)$$

So, $2^{1-a}\int_0^\infty z^{2a-1}e^{-\frac{1}{2}z^2}\,dz = \Gamma(a)$

Immediate consequence of the above is that

$$\Gamma(1/2) = 2^{1-\frac{1}{2}}\int_0^\infty z^{2\frac{1}{2}-1}e^{-\frac{1}{2}z^2}\,dz = \sqrt{2}\int_0^\infty e^{-\frac{1}{2}z^2}\,dz \qquad (*)$$

Therefore, $\left(\Gamma(1/2)\right)^2 = 2\left(\int_0^\infty e^{-\frac{1}{2}z^2}\,dz\right)^2 = 2\int_0^\infty e^{-\frac{1}{2}x^2}\,dx\int_0^\infty e^{-\frac{1}{2}y^2}\,dy$

♦ $\lim_{m\to\infty}\left(-x^{a-1}e^{-x}\Big|_0^m\right) = 0$ after "sufficient" applications of L'Hospital's Rule.

$$= 2\int_0^{\infty}\int_0^{\infty} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}y^2}\,dxdy = 2\int_0^{\infty}\int_0^{\infty} e^{-\frac{1}{2}(x^2+y^2)}\,dxdy$$

Change to polar coordinates by using $x = r\cos\theta,\ y = r\sin\theta,\ dxdy = r dr d\theta$ to obtain

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} re^{-\frac{1}{2}r^2}\,drd\theta = 2\int_{\theta=0}^{\pi/2} d\theta = \pi$$

Therefore, $\left(\Gamma(1/2)\right)^2 = \pi$ so $\Gamma(1/2) = \sqrt{\pi}$!!!

The P. D. F. of the Normal Distribution

Consider the function $n(y) = \dfrac{1}{s\sqrt{2p}} e^{-\frac{1}{2}\left(\frac{y-m}{s}\right)^2}$.

$$\int_{-\infty}^{\infty} n(y)dy = \int_{-\infty}^{\infty} \frac{1}{s\sqrt{2p}} e^{-\frac{1}{2}\left(\frac{y-m}{s}\right)^2} dy \quad \left(\text{Let } z = \frac{y-m}{s},\ dz = \frac{1}{s}dy\right)$$

$$= \frac{1}{\sqrt{2p}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{2}{\sqrt{2p}}\int_0^{\infty} e^{-\frac{1}{2}z^2} dz$$

$$= \frac{1}{\sqrt{p}}\cdot\sqrt{2}\int_0^{\infty} e^{-\frac{1}{2}z^2} dz = \frac{1}{\sqrt{p}}\Gamma\left(1/2\right) \quad \text{(by (*))}$$

$$= \frac{1}{\sqrt{p}}\cdot\sqrt{p} = 1$$

Therefore, $n(y) = \dfrac{1}{s\sqrt{2p}} e^{-\frac{1}{2}\left(\frac{y-m}{s}\right)^2}$ is a valid *probability density function* – the p. d. f. of the ***Normal Distribution***.