

The Variance of the Normal Distribution is σ^2

Let Y be a random variable with p.d.f. $n(y) = \frac{1}{s\sqrt{2p}} e^{-\frac{1}{2}\left(\frac{y-m}{s}\right)^2}$ -- the normal distribution density function.

$$\begin{aligned} E(Y^2) &= \int_{-\infty}^{\infty} y^2 n(y) dy \\ &= \frac{1}{s\sqrt{2p}} \int_{-\infty}^{\infty} y^2 \frac{1}{s\sqrt{p}} e^{-\frac{1}{2}\left(\frac{y-m}{s}\right)^2} dy \\ &= \frac{1}{s\sqrt{2p}} \int_{-\infty}^{\infty} (sz + m)^2 e^{-\frac{1}{2}z^2} s dz \quad \text{where } z = \frac{y-m}{s} \text{ and } dy = s dz \\ &= \frac{1}{\sqrt{2p}} \int_{-\infty}^{\infty} (s^2 z^2 + 2msz + m^2) e^{-\frac{1}{2}z^2} dz \\ &= \frac{1}{\sqrt{2p}} \left[s^2 \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz + 2mp \int_{-\infty}^{\infty} z e^{-\frac{1}{2}z^2} dz + m^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} dz \right] \text{ where the} \end{aligned}$$

limits of integration are understood.

$$\begin{aligned} &= \frac{s^2}{\sqrt{2p}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz + 2mp \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} z e^{-\frac{1}{2}z^2} dz + m^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} e^{-\frac{1}{2}z^2} dz \\ &= \frac{s^2}{\sqrt{2p}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz + 0 + m^2 \cdot 1 \quad \text{since the second integral is the mean} \end{aligned}$$

($m=0$) of the standard normal distribution and the last one is the p.d.f of the standard normal distribution and hence has total area 1 unit.

$$\begin{aligned} &= \frac{s^2}{\sqrt{2p}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2} dz + m^2 \\ &= \frac{s^2}{\sqrt{2p}} \left(-ze^{-\frac{1}{2}z^2} + \int e^{-\frac{1}{2}z^2} dz \right) + m^2 \quad \text{using integration by parts with} \end{aligned}$$

$$\begin{aligned} u &= z & dv &= ze^{-\frac{1}{2}z^2} dz \\ du &= dz & v &= -e^{-\frac{1}{2}z^2} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } E(Y^2) &= \frac{s^2}{\sqrt{2p}} \left[-ze^{-\frac{1}{2}z^2} \right]_{-\infty}^{\infty} + s^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2p}} e^{-\frac{1}{2}z^2} dz + m^2 \cdot 1 \\ &= 0 + s^2 + m^2 \end{aligned}$$

$$\text{So, finally, } V(Y) = E(Y^2) - m^2 = s^2 + m^2 - m^2 = s^2.$$

YAY!

¹ Note: the limit is of the indeterminate form $\infty \cdot 0$ and requires L'Hospital's rule after the proper manipulation.